Electromagnetic Gaussian beams and Riemannian geometry

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Introduction

- A *Gaussian beam* is a solution to Maxwell's equations that look like this:

- A Gaussian beam propagate along a curve:

- **Motivation:** Gaussian beams could (maybe) be used to solve the traveltime problem in EM.

Initial assumptions

- Everything is smooth
- M 3-manifold, bounded/unbounded
- Media is anisotropic, non-homogeneous, no time or frequency dependence
- ε, μ real, positive definite, simultaneously diagonalizable. For some orthogonal matrix R,

$$\varepsilon = R^{-1} \cdot \begin{pmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_3 \end{pmatrix} \cdot R \qquad \mu = R^{-1} \cdot \begin{pmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \mu_3 \end{pmatrix} \cdot R$$

Gaussian beam definition

$$\text{Trial:} \qquad E(x,t) = \mathrm{Re}\{e^{iP\theta(x,t)}E_0(x,t)\}$$

$P > 0$ large constant, E_0 complex 1-form, θ complex phase function

- Trial is a "separation of variables"
- Let $c\colon I \to M$ be a curve, and let

$$\begin{aligned}\phi(t) &= \theta(c(t),t),\\ p_j(t) &= \frac{\partial\theta}{\partial x^j}(c(t),t),\\ H_{ij}(t) &= \frac{\partial^2\theta}{\partial x^i\,\partial x^j}(c(t),t),\end{aligned}$$

Then E is a G.B. on c if $\phi(t), p_i(t)$ are real, and $\mathrm{Im}\, H_{ij}$ is positive definite.

Basic property of Gaussian beams

Let us fix t, and let

$$z = z(x) = x - c(t)$$

then by Taylor's theorem,

$$\theta(x(z), t) \quad = \quad \phi(t) \quad + \quad p_i(t) \cdot z^i \quad + \quad \frac{1}{2} H_{ij}(t) z^i z^j + \cdots$$

Then

$$\begin{aligned} |\exp(iP\theta)| \quad &= \quad \exp\left(-\frac{P}{2}\ z^T \cdot \mathrm{Im}\, H \cdot z\right) \\ &= \quad \text{Gaussian bell curve} \end{aligned}$$

Propagating Gaussian beams

- *Hamilton-Jacobi* equation for θ: $\frac{\partial\theta}{\partial t} = h(d\theta)$ where Hamiltonian $h\colon T^*M \to \mathbb{R}$ ($h = h_\pm$ depend only on media)

- Expanding both sides as a Taylor series at $z = 0$ gives sufficient condition on c, ϕ, p, H:

  – ϕ is constant

  – (c, p) is a solution to Hamilton's equations

$$\begin{aligned} \frac{dc^i}{dt} &= \frac{\partial h}{\partial \xi_i} \circ (c, p), \\ \frac{dp_i}{dt} &= -\frac{\partial h}{\partial x^i} \circ (c, p). \end{aligned}$$

  – H is a solution to a matrix Riccati equation (depending on h)

Geometry of TM vs. geometry of T^*M

TM

- On TM, a curve $c\colon I \to M$ has a canonical lift, $\hat{c}\colon I \to TM$.
- "Traditional" geometry (metric tensor, curvature tensors, etc.) exist on TM.
- Finsler norm $F\colon TM \to \mathbb{R}$, (in Riemannian case: $F(y) = \sqrt{g_{ij}(x)y^i y^j}, y \in T_x M$)
  $\to$ Geodesic equation $\to$ Geodesics

T^*M

- T^*M has a canonical symplectic structure.
- Hamiltonian function: $h\colon T^*M \to \mathbb{R}$
  $\to$ Hamilton equations $\to$ bicharacteristics

Legendre transformation

- In mechanics: Hamilton eqs. $\leftrightarrow$ Euler-Lagrange eqs.
- Mathematical interpretation: Legedre transformation: Symplectic geometry $\leftrightarrow$ Riemann-Finsler geometry
- **Legendre transformation**: bijection $L\colon T^*M \to TM$ that preserves structure. Examples:
  - h strongly convex, 1-homogeneous $\longrightarrow$ Finsler geometry
  - $h = \sqrt{\text{pos. def. quadratic form}} \longrightarrow$ Riemann geometry

Legendre transformation for quadratic Hamiltonians

- Suppose $h\colon T^*M \to \mathbb{R}$ has the form

$$h(x,\xi) = \sqrt{h^{ij}(x)\xi_i\xi_j}.$$

where h^{ij} is positive definite for each x.

$$\Rightarrow \qquad \begin{aligned} L\colon T^*M &\to TM, \\ \xi_i dx^i &\mapsto h^{ij}\xi_j \frac{\partial}{\partial x^j} \end{aligned}$$

is a bijection, and

$$\|y\| = h \circ L^{-1}(y) = \sqrt{h_{ij}(x)y^i y^j}, \quad h_{ij} = (h^{ij})^{-1}_{ij}$$

is a norm induced by a Riemannian inner product $g_{ij} = h_{ij}$.

- (c,p) is a bicharacteristic w.r.t. $h \Rightarrow c$ is geodesic w.r.t. g.
- c is geodesic w.r.t. $g \Rightarrow L^{-1} \circ \hat{c}$ is a bicharacteristic w.r.t. h

Hamiltonians $h_\pm$

- For $(x,\xi) \in T^*M$, let

$$M(x,\xi) = \begin{pmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \mu_3 \end{pmatrix} \cdot \begin{pmatrix} & \xi \times I \\ -\xi \times I & \end{pmatrix}$$

- $h_\pm \colon T^*M \to \mathbb{R}$ are continuous functions such that

$$\sigma(M(x,\xi)) = \pm\{0, h_+(x,\xi), h_-(x,\xi)\},$$

and $h_\pm(x,\xi) \geq 0$.

- $\sigma(M(x,\xi))$ is coordinate invariant, but $h_\pm$ are not well defined.

Example: In isotropic media, $\varepsilon = \varepsilon(x)I$, $\mu = \mu(x)I$,

$$h_\pm(x,z) = \frac{1}{\sqrt{\varepsilon(x)\mu(x)}}\|z\|.$$

Computer algebra: If $(x,\xi) \in T^*M$ and $\eta = \det R\; R \cdot \xi$, then $\lambda \in \sigma(M(x,\xi))$ if and only if

$$\lambda^2\big(\lambda^4 - \|S \cdot \eta\|^2\, \lambda^2 + \|F \cdot \eta\|^2\; \|N \cdot \eta\|^2\big) = 0,$$

where

$$\begin{aligned}
e_i &= 1/\sqrt{\varepsilon_i}, \quad m_i = 1/\sqrt{\mu_i}, \\
S &= \operatorname{diag}\left(\sqrt{e_2^2 m_3^2 + e_3^2 m_2^2}, \sqrt{e_1^2 m_3^2 + e_3^2 m_1^2}, \sqrt{e_1^2 m_2^2 + e_2^2 m_1^2}\right), \\
F &= \operatorname{diag}\,(e_2 e_3, e_1 e_3, e_1 e_2), \\
N &= \operatorname{diag}\,(m_2 m_3, m_1 m_3, m_1 m_2),
\end{aligned}$$

$\sqrt{\cdot}$ is the pos. square root,
$\| \cdot \|$ Euclidean norm.

- Quadratic solution formula:

$$\sigma(M(x,\xi)) = \pm\left\{0, \frac{1}{\sqrt{2}}\sqrt{\|S\cdot\eta\|^2 \pm \sqrt{D(x,\xi)}}\right\}, \tag{1}$$

where $\pm$-signs are independent and D is the 4-homogeneous polynomial

$$D(x,\xi) = \|S\cdot\eta\|^4 - 4\|F\cdot\eta\|^2\ \|N\cdot\eta\|^2.$$

- This gives

$$h_\pm(x,\xi) = \frac{1}{\sqrt{2}}\sqrt{\|S\cdot\eta\|^2 \pm \sqrt{D(x,\xi)}}.$$

The Δ_{ij}-symbols

- For $i, j = 1, 2, 3$ let

$$\Delta_{ij} = e_i^2 m_j^2 - e_j^2 m_i^2.$$

- If two Δ_{ij}-symbols vanish, then all symbols vanish.
- 3 possibilities: all $\Delta_{ij} = 0$, one $\Delta_{ij} = 0$, none $\Delta_{ij} = 0$.
- $h_\pm$-functions behave qualitatively differently depending on how many of the Δ_{ij}-symbols vanish.
- Computer algebra:

$$\begin{aligned} D &= (\Delta_{23}\xi_1^2 + \Delta_{13}\xi_2^2 + \Delta_{12}\xi_3^2)^2 - 4\Delta_{12}\Delta_{23}\xi_1^2\xi_3^2 \\ &= (\Delta_{23}\xi_1^2 + \Delta_{13}\xi_2^2 + \Delta_{21}\xi_3^2)^2 - 4\Delta_{21}\Delta_{13}\xi_2^2\xi_3^2 \\ &= (\Delta_{32}\xi_1^2 + \Delta_{13}\xi_2^2 + \Delta_{12}\xi_3^2)^2 - 4\Delta_{13}\Delta_{32}\xi_1^2\xi_2^2. \end{aligned}$$

The three media classes

Proposition 0.1 (Characterization of media I, Δ_{ij} all zero)
The following are equivalent:

1. *All Δ_{ij}-symbols vanish.*
2. *At least two Δ_{ij}-symbols vanish.*
3. $h_+ = h_-$
4. *The medium matrices satisfy*

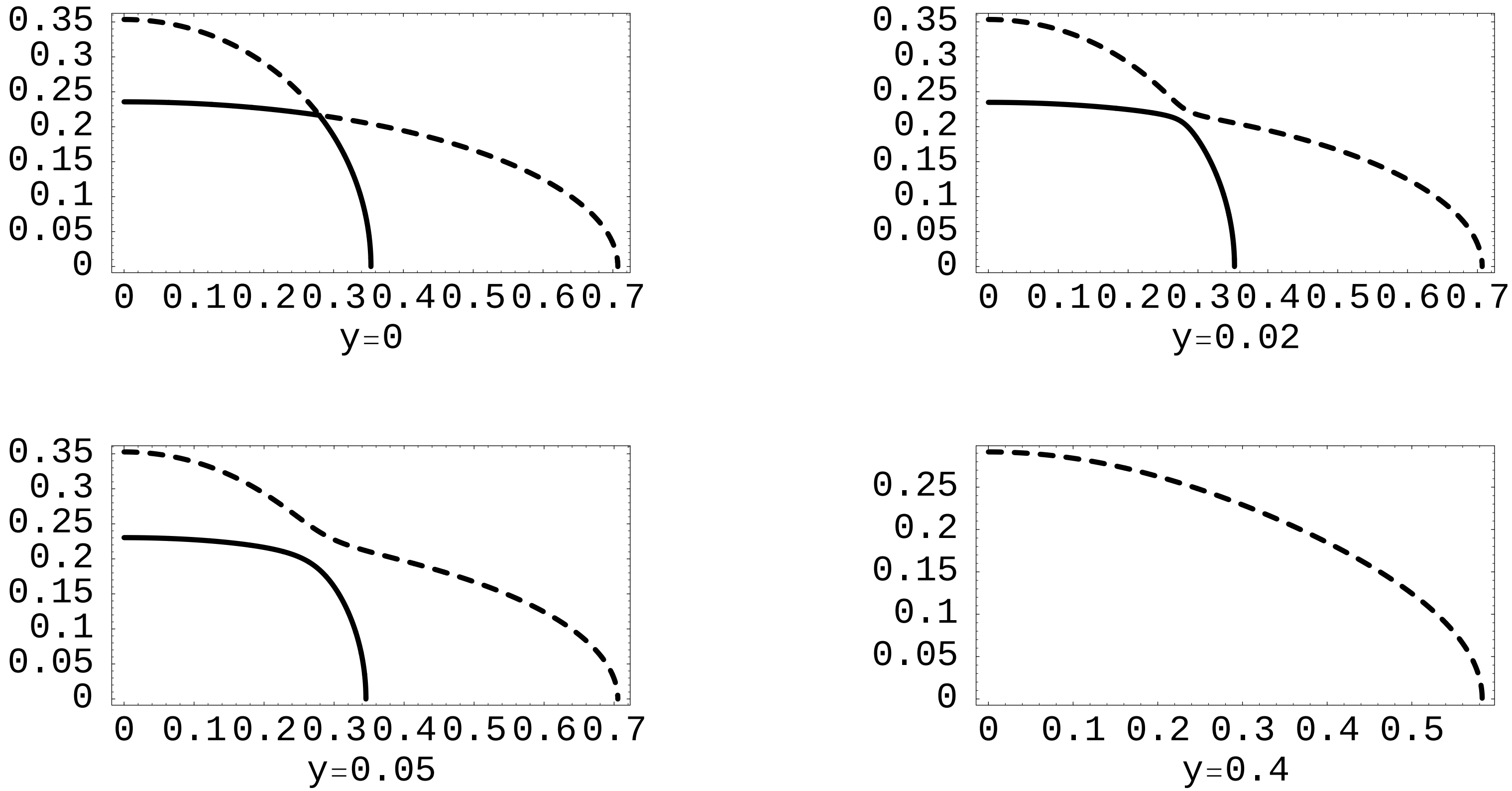
$$\varepsilon = \rho\mu$$

for some $\rho > 0$.

Proposition 0.2 (Characterization of media II) *The following are equivalent:*

1. *At least one of the* Δ_{ij}*-symbols vanishes.*
2. $h_\pm = \sqrt{\textit{pos. def. quadratic form}}$
3. $h_\pm$ *are smooth on* $\mathbb{R}^3 \setminus \{0\}$.
4. h_- *is convex on* $\mathbb{R}^3$.

Characterization of media III: All $\Delta_{ij} \neq 0$



Examples

$\Delta_{23} = 0$, that is, $\varepsilon_3\mu_2 = \varepsilon_2\mu_3$

$$\begin{aligned} g_{+,ij}(x) &= (R^{-1} \cdot \text{diag}\,(\varepsilon_3\mu_2, \varepsilon_1\mu_3, \varepsilon_1\mu_2) \cdot R)_{ij} \\ g_{-,ij}(x) &= (R^{-1} \cdot \text{diag}\,(\varepsilon_3\mu_2, \varepsilon_3\mu_1, \varepsilon_2\mu_1) \cdot R)_{ij} \end{aligned}$$

Media: $(\varepsilon_1, \varepsilon_2, \varepsilon_2)$, (μ_1, μ_2, μ_2),

$$\begin{aligned} g_{+,ij}(x) &= \mu_2(R^{-1} \cdot \text{diag}\,(\varepsilon_2, \varepsilon_1, \varepsilon_1) \cdot R)_{ij} \\ g_{-,ij}(x) &= \varepsilon_2(R^{-1} \cdot \text{diag}\,(\mu_2, \mu_1, \mu_1) \cdot R)_{ij} \end{aligned}$$

Media: $(\varepsilon_1, \varepsilon_2, \varepsilon_2)$, (μ_1, μ_1, μ_1),

$$\begin{aligned} g_{+,ij}(x) &= \mu_2(R^{-1} \cdot \text{diag}\,(\varepsilon_2, \varepsilon_1, \varepsilon_1) \cdot R)_{ij} \\ g_{-,ij}(x) &= \varepsilon_2\mu_2\delta_{ij} \end{aligned}$$

Media: $(\varepsilon_1, \varepsilon_1, \varepsilon_1)$, (μ_1, μ_1, μ_1),

$$g_{\pm,ij} = \varepsilon_1\mu_1\delta_{ij}$$