Classification of electromagnetic media by behaviour of phase velocity

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Problem statement

- Two descriptions of homogeneous anisotropic electromagnetic medium
  - Analytic: Coefficients in Maxwell's equations.
  - Geometric/Dynamic: Phase velocity of plane waves.

- **Questions:**
  - How are these related?
  - Two definitions of isotropic medium?

Maxwell Equations

- **Base space:** $\mathbb{R}^4$ (or arbitrary 4-manifold)
- **Electromagnetic fields:** $F, G \in \Omega^2(\mathbb{R}^4)$.
- **Sourceless Maxwell's equations:**

$$\begin{aligned} dF &= 0, \\ dG &= 0. \end{aligned}$$

- **Model for medium — Constitutive equation:**

$$G = \kappa(F)$$

where $\kappa \in \Omega^2_2(\mathbb{R}^4)$ is an antisymmetric $\binom{2}{2}$-tensor

$$\kappa\colon \Omega^2(\mathbb{R}^4) \to \Omega^2(\mathbb{R}^4)$$

- **Non-dissipative medium:** for all $u, v \in \Omega^2(\mathbb{R}^4)$,

$$\kappa(u) \wedge v = u \wedge \kappa(v)$$

Back to $\mathbb{R}^3$

- $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3, \quad F = B + E \wedge dt, \quad G = D - H \wedge dt.$
- **Sourceless Maxwell's equations:**

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & \nabla \cdot \mathbf{D} &= 0, \\ \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t}, & \nabla \cdot \mathbf{B} &= 0. \end{aligned}$$

- **Model for medium:** $\begin{pmatrix} \mathbf{H} \\ \mathbf{D} \end{pmatrix} = \begin{pmatrix} \mathscr{C} & \mathscr{B} \\ \mathscr{A} & \mathscr{D} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}$
- **Non-dissipative:** $\mathscr{A} = \mathscr{A}^t$, $\mathscr{B} = \mathscr{B}^t$, $\mathscr{C} + \mathscr{D}^t = 0$.
  κ is determined by 21 real numbers.
- **Poynting's theorem:** If κ does not depend on t,

$$\frac{d}{dt} \int_U \frac{1}{2} (E \wedge D + H \wedge B) = -\int_{\partial U} E \wedge H.$$

Fresnel surface

- **Plane wave solution:** (κ constant coefficient)

$$F = \operatorname{Re}\left\{ e^{i\Phi}\alpha \right\}, \quad G = \operatorname{Re}\left\{ e^{i\Phi}\beta \right\}.$$

- $dG = 0, G = \kappa(F)$ implies: $d\Phi \wedge \beta = 0, \quad \beta = \kappa(\alpha)$. Thus.

$$(d\Phi \wedge \kappa)(\alpha) = 0$$

This has a solution $\alpha \neq 0$ if and only if $p(d\Phi) = 0$ where p is the **Fresnel polynomial** $p(\xi) = \mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l$ for

$$\mathscr{G}^{ijkl} = \kappa_{ab}^{pq}\kappa_{cd}^{ri}\kappa_{ef}^{sj}\varepsilon^{abek}\varepsilon^{cdfl}\varepsilon_{pqrs}.$$

- $\{\xi \in T^*\mathbb{R}^4 : p(\xi) = 0\}$ is the **Fresnel surface** $F(\kappa)$.
- Obukhov, Fukui, Rubilar (2000).

Invariances of Fresnel surface $F(\kappa)$

- For any κ and $\alpha \neq 0$,

$$\begin{aligned} F(\alpha\kappa) &= F(\kappa), \\ F(\kappa) &= F(\kappa + \alpha\,\mathrm{Id}). \end{aligned}$$

- For any invertible κ,

$$F(\kappa) = F(\kappa^{-1}).$$

- All invariances are not known.

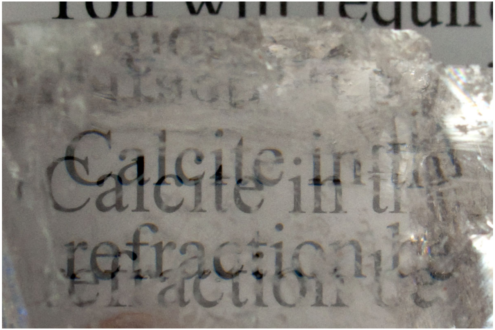
Example:

- **Isotropic medium:** $\kappa = \lambda *_g$ for Lorentz metric.

$$F(\kappa) \;=\; \text{Lorentz null cone of } g.$$

- **Uniaxial medium.** $\epsilon = \operatorname{diag}(\epsilon_1, \epsilon_2, \epsilon_2)$, $\mu > 0$.

$$F(\kappa) \;=\; \text{union of two Lorentz null cones}$$



Iceland spar (Image source: Wikipedia)

Classification of medium with two Lorentz null cones

- **Theorem: [D.]** *Suppose* $\kappa \in \Omega^2_2(\mathbb{R}^4)$ *is constant coefficient. Furthermore, suppose that*

  (i) κ *is non-dissipative,*

  (ii) κ *is invertible,*

  (iii) $F(\kappa) =$ union of two Lorentz null cones.

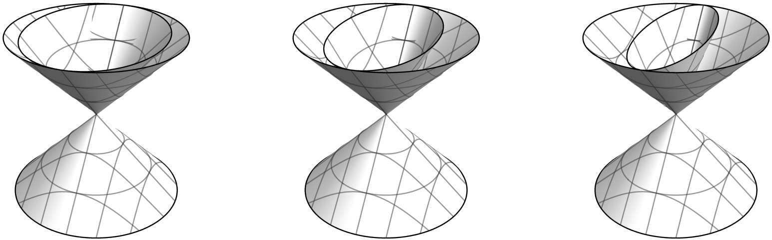
  Then there only three possibilities:

Possibility 1 of 3: Uniaxial-type media

- Constitutive equation: $[\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R},\ \text{sgn}\, \beta_1 = \text{sgn}\, \beta_2 \neq 0]$

$$\begin{pmatrix} \mathbf{H} \\ \mathbf{D} \end{pmatrix} = \left(\begin{array}{ccc|ccc} \alpha_1 & & & \beta_1 & & \\ & \alpha_2 & & & \beta_2 & \\ & & \alpha_2 & & & \beta_2 \\ \hline \beta_1 & & & -\alpha_1 & & \\ & \beta_2 & & & -\alpha_2 & \\ & & \beta_2 & & & -\alpha_2 \end{array}\right) \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}$$

- Relative configuration of null cones:

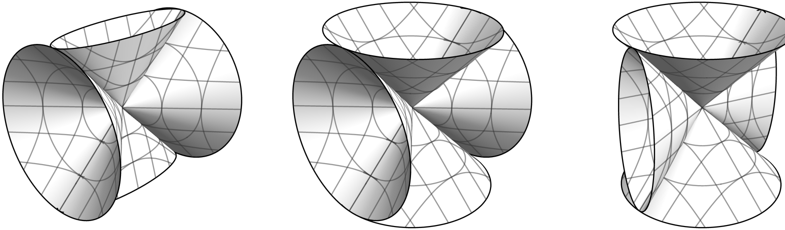


Possibility 2 of 3:

- Constitutive equation: $[\alpha_1, \alpha_2 \in \mathbb{R},\ \beta_1, \beta_2 \in \mathbb{R} \setminus \{0\}]$

$$\begin{pmatrix} \mathbf{H} \\ \mathbf{D} \end{pmatrix} = \left(\begin{array}{ccc|ccc} \alpha_1 & & & \beta_1 & & \\ & \alpha_1 & & & \beta_2 & \\ & & \alpha_2 & & & \beta_2 \\ \hline {\color{blue}-\beta_1} & & & -\alpha_1 & & \\ & \beta_2 & & & -\alpha_2 & \\ & & \beta_2 & & & -\alpha_2 \end{array}\right) \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}$$

- Relative configuration of null cones:

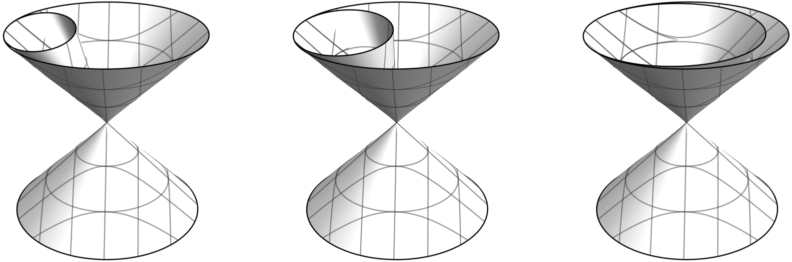


Possibility 3 of 3:

- Constitutive equation: $[\alpha \in \mathbb{R}, \beta \in \mathbb{R} \setminus \{0\},\ w = \sqrt{1+4\beta^2}]$

$$
\begin{pmatrix} \mathbf{H} \\ \mathbf{D} \end{pmatrix} = \frac{\beta}{w} \left(\begin{array}{ccc|ccc}
\alpha & & & -\frac{w^2}{\beta} & & \\
 & \alpha & {\color{blue}2} & & -w-2 & \\
 & & \alpha & & & -w \\
\hline
-\beta & & & -\alpha & & \\
 & -w & & & -\alpha & \\
 & & -w+2 & & {\color{blue}-2} & -\alpha
\end{array}\right) \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}
$$

- Relative configuration of null cones:



Outline of proof: (in theory)

1. Factorisation:

$$\left(\kappa_{ab}^{pq}\kappa_{cd}^{ri}\kappa_{ef}^{sj}\varepsilon^{abek}\varepsilon^{cdfl}\varepsilon_{pqrs}\right)\xi_i\xi_j\xi_k\xi_l = (g^{ij}\xi_i\xi_j)(h^{kl}\xi_k\xi_l)$$

2. Identifying coefficients gives 35 polynomial equations:

$$P_k(\kappa, \quad g, \quad h) = 0, \quad k \in \{1, \dots, 35\}$$

3. Eliminate variables in g and h

$$Q_k(\kappa) = 0, \quad k \in \{1, \dots, N\} \qquad\qquad (*)$$

4. Solve all κ that satisfy equation $(*)$.
   - Include solutions where g, h are Lorentz.
   - Exclude solutions with other signatures, complex g, h, etc.

Outline of proof: (in practice)

(a) Eliminating variables in polynomial systems can be done with *Gröbner bases*, but is computationally expensive (35 eqs, 21+10+10=41 variables, 3rd order).

(b) Simplify κ by a Jordan normal form.

- Idea: κ can be represented by 6×6 matrix K.
- $S \cdot K \cdot S^{-1}=$ Jordan block form.
- This can be done by a coordinate transformation (+ simple operators) (Schuller, Witte, Wohlfarth, Annals of Physics **325**, 2010)
- $\rightarrow$ Non-dissipative media has 23 normal forms:
- Case by case analysis of normal forms $1, \ldots, 7$.
- Exclude normal forms 8, $\ldots$, 23 by general results from Schuller et al.

Classification of medium with one Lorentz null cone

- **Theorem: [D.]** *Suppose* $\kappa \in \Omega^2_2(\mathbb{R}^4)$ *is constant coefficient. Furthermore, suppose that*

  (i) κ *is non-dissipative,*

  (ii) κ *is invertible,*

  (iii) $\operatorname{trace} \kappa = 0$

  (iv) $F(\kappa) =$ Lorentz null cone of a Lorentz metric g.

  Then there exists a $\lambda \in \mathbb{R} \setminus \{0\}$ such that

$$\kappa = \lambda *_g .$$

Classification of isotropic media

- Previous results:
  - [Obukhov, Fukui, Rubilar, 2000] Case $\mathscr{C} = 0$.
  - [Favaro, Bergamin, 2011] κ is isotropic if $p(\xi) = C(g^{ij}\xi_i\xi_j)^2$ for a g with Lorentz signature.
- For related results, see also: [Obukhov, Rubilar, 2002], [Hehl, Lämmerzahl, 2004], [Itin, 2005]

Summary

- Isotropic medium can recognised from $F(\kappa)$ (under mild assumptions).
- Complete classification of medium with two Lorentz null cones.
  - More complicated.
  - Two new medium classes.
- One can never uniquely determine κ from $F(\kappa)$. There are always some invariances. To determine κ one needs more information (like polarisation).
- Here we have worked with constant coeffient media $\mathbb{R}^4$. The results also generalise to arbitrary medium tensors on an arbitrary 4-manifold, but then the results is pointwise.

Thank you!

Possibility 3 of 3: one, two, or three phase velocities