Electromagnetic media with two Lorentz null cones

Matias Dahl

Aalto University, Finland

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Problem statement

- Two descriptions of homogeneous anisotropic electromagnetic medium
  - Analytic: Coefficients in Maxwell's equations.
  - Geometric: Phase velocity of plane waves.
- How are these related?

Maxwell Equations

- **Base space:** $\mathbb{R}^4$
- **Electromagnetic fields:** $F, G \in \Omega^2(\mathbb{R}^4)$.
- **Sourceless Maxwell's equations:**

$$\begin{aligned} dF &= 0, \\ dG &= 0. \end{aligned}$$

- **Model for medium — Constitutive equation:**

$$G = \kappa(F)$$

  where $\kappa \in \Omega^2_2(\mathbb{R}^4)$ is an antisymmetric $\binom{2}{2}$-tensor with constant coefficients

$$\kappa\colon \Omega^2(\mathbb{R}^4) \to \Omega^2(\mathbb{R}^4)$$

- **Non-dissipative medium:** for all $u, v \in \Omega^2(\mathbb{R}^4)$,

$$\kappa(u) \wedge v = u \wedge \kappa(v)$$

Back to $\mathbb{R}^3$

- $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3, \quad F = B + E \wedge dt, \qquad G = D - H \wedge dt.$
- **Sourceless Maxwell's equations:**

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & \nabla \cdot \mathbf{D} &= 0, \\ \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t}, & \nabla \cdot \mathbf{B} &= 0. \end{aligned}$$

- **Model for medium:** $\begin{pmatrix} \mathbf{H} \\ \mathbf{D} \end{pmatrix} = \begin{pmatrix} C & B \\ A & D \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}$
- **Non-dissipative:** $A = A^t$, $B = B^t$, $C + D^t = 0$.
  κ is determined by 21 real numbers.
- **Poynting's theorem:**

$$\frac{d}{dt} \int_U \frac{1}{2} (E \wedge D + H \wedge B) = - \int_{\partial U} E \wedge H.$$

Characteristic polynomial I

- **Plane wave solution:**

$$F = \mathrm{Re}\left\{e^{i\Phi}A\right\}, \quad G = \mathrm{Re}\left\{e^{i\Phi}B\right\}.$$

- $dG = 0, G = \kappa(F)$ implies: $d\Phi \wedge B = 0, \quad B = \kappa(A)$. Thus.

$$(d\Phi \wedge \kappa)(A) = 0$$

  This has a solution $A \neq 0$ if and only if $p(d\Phi) = 0$ where p is the **characteristic polynomial** $p(\xi) = \mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l$ for

$$\mathscr{G}^{ijkl} = \kappa_{ab}^{pq}\kappa_{cd}^{ri}\kappa_{ef}^{sj}\varepsilon^{abek}\varepsilon^{cdfl}\varepsilon_{pqrs}.$$

- $\{\xi \in T^*\mathbb{R}^4 : p(\xi) = 0\}$ is the *Fresnel surface* $F(\kappa)$.
- Obukhov, Fukui, Rubilar (2000).

Characteristic polynomial II

- **Evolution eqs:** $L^{(0)} \frac{\partial}{\partial x^0} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} + \sum_{k=1}^{3} L^{(k)} \frac{\partial}{\partial x^k} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = 0.$
- **Coefficient matrices:** $L^{(0)}, \ldots, L^{(3)} \in \mathbb{R}^{6\times 6}$

$$L^{(0)} = \begin{pmatrix} A & D \\ 0 & \mathrm{Id} \end{pmatrix}, \quad L^{(k)} = \begin{pmatrix} (\varepsilon^{ijk})_{i,j=1}^3 & 0 \\ 0 & (\varepsilon^{ijk})_{i,j=1}^3 \end{pmatrix} \begin{pmatrix} C & B \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

- Then

$$p(\xi) = \det\left(\xi_0 L^{(0)} + \sum_{k=1}^{3} \xi_k L^{(k)} \right) / \xi_0^2.$$

- Schuller, Witte, Wohlfarth (2010).

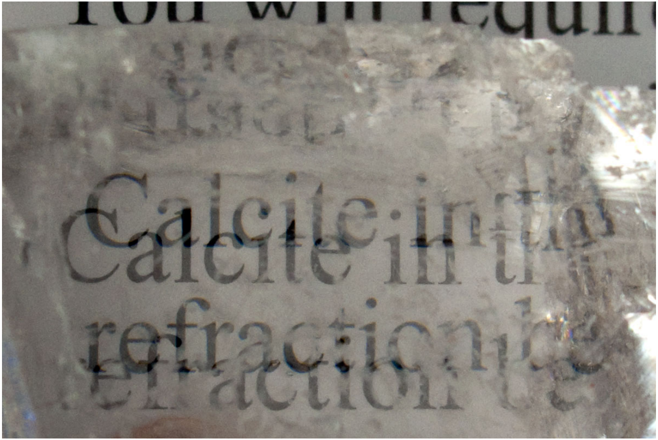
Possible factorisations for $p(\xi)$

- One of the below holds:
  - **Case 1:** $p = 0$.
  - **Case 2:** p has a linear factor.
  - **Case 3:** $p = \lambda(g^{ij}\xi_i\xi_j)^2$ for irreducible $g^{ij}\xi_i\xi_j$.
    - Example: isotropic media, $\kappa = f *_g$, g Lorentz
    - Also characterisation (Favaro, Bergamin, Annalen der Physik **523**, 2011).
  - **Case 4:** $p = (g^{ij}\xi_i\xi_j)(h^{ij}\xi_i\xi_j)$ with both factors irreducible and non-proportional.
    - Example: uniaxial media.
  - **Case 5:** p is irreducible.
    - Example: biaxial media.

Example:

- **Uniaxial medium.** $\epsilon = \mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_2)$, $\mu > 0$.

  $F(\kappa)$= union of two Lorentz null cones



Iceland spar (Image source: Wikipedia)

- **Theorem: [D.]** *Suppose* $\kappa \in \Omega^2_2(\mathbb{R}^4)$ *is constant coefficient. Furthermore, suppose that*

  (i) κ *is non-dissipative,*

  (ii) κ *is invertible,*

  (iii) $F(\kappa) =$ union of two Lorentz null cones.

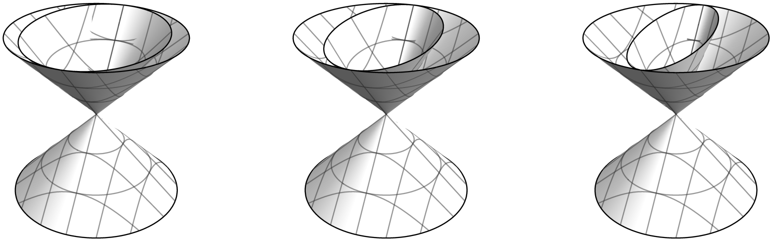
  Then there only three possibilities:

Possibility 1 of 3: Uniaxial-type media

- Constitutive equation: $[\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R},\ \operatorname{sgn}\beta_1 = \operatorname{sgn}\beta_2 \neq 0]$

$$\begin{pmatrix} \mathbf{H} \\ \mathbf{D} \end{pmatrix} = \left(\begin{array}{ccc|ccc} \alpha_1 & & & \beta_1 & & \\ & \alpha_2 & & & \beta_2 & \\ & & \alpha_2 & & & \beta_2 \\ \hline \beta_1 & & & -\alpha_1 & & \\ & \beta_2 & & & -\alpha_2 & \\ & & \beta_2 & & & -\alpha_2 \end{array}\right) \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}$$

- Relative configuration of null cones:

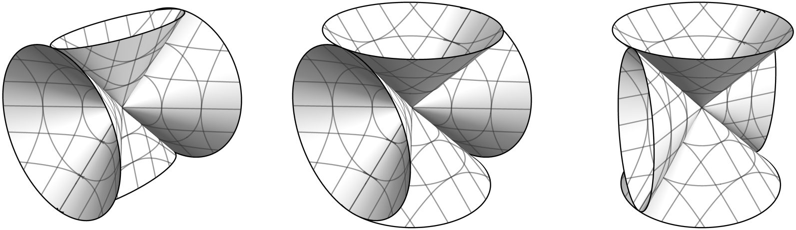


Possibility 2 of 3:

- Constitutive equation: $[\alpha_1, \alpha_2 \in \mathbb{R},\ \beta_1, \beta_2 \in \mathbb{R} \setminus \{0\}]$

$$\begin{pmatrix} \mathbf{H} \\ \mathbf{D} \end{pmatrix} = \left(\begin{array}{ccc|ccc} \alpha_1 & & & \beta_1 & & \\ & \alpha_1 & & & \beta_2 & \\ & & \alpha_2 & & & \beta_2 \\ \hline -\beta_1 & & & -\alpha_1 & & \\ & \beta_2 & & & -\alpha_2 & \\ & & \beta_2 & & & -\alpha_2 \end{array}\right) \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}$$

- Relative configuration of null cones:

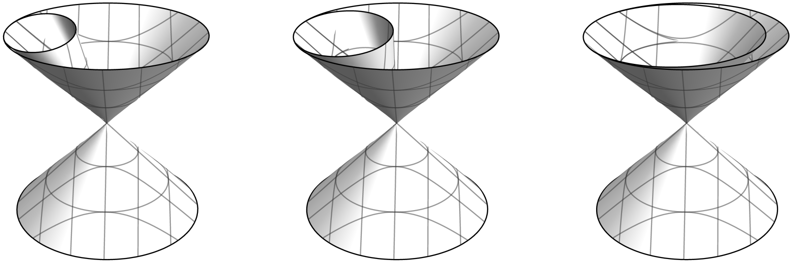


Possibility 3 of 3:

- Constitutive equation: $[\alpha \in \mathbb{R}, \beta \in \mathbb{R} \setminus \{0\},\ w = \sqrt{1 + 4\beta^2}]$

$$\begin{pmatrix} \mathbf{H} \\ \mathbf{D} \end{pmatrix} = \frac{\beta}{w} \left(\begin{array}{ccc|ccc} \alpha & & & -\frac{w^2}{\beta} & & \\ & \alpha & 2 & & -w-2 & \\ & & \alpha & & & -w \\ \hline -\beta & & & -\alpha & & \\ & -w & & & -\alpha & \\ & & -w+2 & & -2 & -\alpha \end{array} \right) \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}$$

- Relative configuration of null cones:



Outline of proof: (in theory)

1. Factorisation:

$$\left(\kappa^{pq}_{ab}\kappa^{ri}_{cd}\kappa^{sj}_{ef}\varepsilon^{abek}\varepsilon^{cdfl}\varepsilon_{pqrs}\right)\xi_i\xi_j\xi_k\xi_l = (g^{ij}\xi_i\xi_j)(h^{kl}\xi_k\xi_l)$$

2. Identifying coefficients gives 35 polynomial equations:

$$P_k(\kappa, \quad g, \quad h) = 0, \quad k \in \{1, \dots, 35\}$$

3. Eliminate variables in g and h

$$Q_k(\kappa) = 0, \quad k \in \{1, \dots, N\} \qquad (*)$$

4. Solve all κ that satisfy equation $(*)$.
   - Include solutions where g, h are Lorentz.
   - Exclude solutions with other signatures, complex g, h, etc.

Outline of proof: (in practice)

(a) Eliminating variables in polynomial systems can be done with *Gröbner bases*, but is computationally expensive (35 eqs, 21+10+10=41 variables, 3rd order).

(b) Simplify κ by a Jordan normal form.

- Idea: κ can be represented by 6×6 matrix K.
- $S \cdot K \cdot S^{-1}$= Jordan block form.
- This can be done by a coordinate transformation (+ simple operators) (Schuller, Witte, Wohlfarth, Annals of Physics **325**, 2010)
- $\rightarrow$ Non-dissipative media has 23 normal forms:
- Case by case analysis of normal forms $1, \ldots, 7$.
- Exclude normal forms 8, $\ldots$, 23 by general results from Schuller et al.

Thank you!

Possibility 3 of 3: one, two, or three phase velocities