In[1]:=
```
SetDirectory["~/KappaLib/"];
<< kappaLib-1.2.m
<< helper.m
```

```
Loading KappaLib v1.2

Loading helper.m..
```

- **Suppose G^ijkl is a symmetric tensor and**

  { xi in R^4 : gamma(xi) = 0 } supset { xi in R^4 : g(xi,xi) = 0 }

  for gamma(xi) = G^ijkl xi_i xi_j xi_k xi_l,

  g = diag(-1,1,1,1).

  Claim: There exists a quadratic form h such that

  gamma = g(xi,xi) h(xi,xi) for all xi in R^4

- **First we define g = diag(-1,1,1,1) and gamma**

In[4]:=
```
g = DiagonalMatrix[{-1, 1, 1, 1}];
```

In[5]:=
```
symm[i_, j_, k_, l_] := Module[{tmp},
  tmp = Sort[{i, j, k, l}];
  ToExpression[
   "s"
    <> ToString[tmp[[1]]]
    <> ToString[tmp[[2]]]
    <> ToString[tmp[[3]]]
    <> ToString[tmp[[4]]]
  ]
 ]
```

In[6]:=
```
symm[1, 3, 2, 2]
symm[4, 3, 2, 1]
```

Out[6]= s1223

Out[7]= s1234

In[8]:=
```
(* define gamma as above *)
gamma[v_] := Sum[
  symm[i, j, k, l] v[[i]] v[[j]] v[[k]] v[[l]],
  {i, 1, 4},
  {j, 1, 4},
  {k, 1, 4},
  {l, 1, 4}
 ]

(* define quadratic form g *)
G[v_] := v.g.v
```

In[10]:=
```
gamma[{1, 1, 0, 0}]
G[{1, 1, 0, 0}]
```

Out[10]= s1111 + 4 s1112 + 6 s1122 + 4 s1222 + s2222

Out[11]= 0

Find null vectors for G.

Note: We do not need to trust the below code. We will later check that the vectors are indeed null vectors for G.

In[12]:=
```
coordValues = {1, -1, Sqrt[2], -Sqrt[2], Sqrt[3], -Sqrt[3], 0};

LL = Length[coordValues];

nullVectors = {};

For[i0 = 1, i0 ≤ LL, i0++,
 For[i1 = 1, i1 ≤ LL, i1++,
  For[i2 = 1, i2 ≤ LL, i2++,
   For[i3 = 1, i3 ≤ LL, i3++,

    frPoint = {
      coordValues[[i0]],
      coordValues[[i1]],
      coordValues[[i2]],
      coordValues[[i3]]};

    (*frPoint=vars/.frSub;*)

    (* if point is non-zero and belongs
      to the Fresnel surface add it to list. *)
    If[Simplify[frPoint.frPoint] ≠ 0,
     If[Simplify[G[frPoint] == 0],
       nullVectors = Append[nullVectors, frPoint];
      ];
    ];

   ]
  ]
 ]
]
```

In[16]:=
```
nullVectors
```

Out[16]=
```
{{1, 1, 0, 0}, {1, -1, 0, 0}, {1, 0, 1, 0}, {1, 0, -1, 0}, {1, 0, 0, 1}, {1, 0, 0, -1},
 {-1, 1, 0, 0}, {-1, -1, 0, 0}, {-1, 0, 1, 0}, {-1, 0, -1, 0}, {-1, 0, 0, 1},
 {-1, 0, 0, -1}, {√2, 1, 1, 0}, {√2, 1, -1, 0}, {√2, 1, 0, 1}, {√2, 1, 0, -1},
 {√2, -1, 1, 0}, {√2, -1, -1, 0}, {√2, -1, 0, 1}, {√2, -1, 0, -1}, {√2, √2, 0, 0},
 {√2, -√2, 0, 0}, {√2, 0, 1, 1}, {√2, 0, 1, -1}, {√2, 0, -1, 1}, {√2, 0, -1, -1},
 {√2, 0, √2, 0}, {√2, 0, -√2, 0}, {√2, 0, 0, √2}, {√2, 0, 0, -√2},
 {-√2, 1, 1, 0}, {-√2, 1, -1, 0}, {-√2, 1, 0, 1}, {-√2, 1, 0, -1}, {-√2, -1, 1, 0},
 {-√2, -1, -1, 0}, {-√2, -1, 0, 1}, {-√2, -1, 0, -1}, {-√2, √2, 0, 0},
 {-√2, -√2, 0, 0}, {-√2, 0, 1, 1}, {-√2, 0, 1, -1}, {-√2, 0, -1, 1},
 {-√2, 0, -1, -1}, {-√2, 0, √2, 0}, {-√2, 0, -√2, 0}, {-√2, 0, 0, √2},
 {-√2, 0, 0, -√2}, {√3, 1, 1, 1}, {√3, 1, 1, -1}, {√3, 1, -1, 1}, {√3, 1, -1, -1},
 {√3, 1, √2, 0}, {√3, 1, -√2, 0}, {√3, 1, 0, √2}, {√3, 1, 0, -√2},
 {√3, -1, 1, 1}, {√3, -1, 1, -1}, {√3, -1, -1, 1}, {√3, -1, -1, -1},
 {√3, -1, √2, 0}, {√3, -1, -√2, 0}, {√3, -1, 0, √2}, {√3, -1, 0, -√2},
 {√3, √2, 1, 0}, {√3, √2, -1, 0}, {√3, √2, 0, 1}, {√3, √2, 0, -1},
 {√3, -√2, 1, 0}, {√3, -√2, -1, 0}, {√3, -√2, 0, 1}, {√3, -√2, 0, -1},
 {√3, √3, 0, 0}, {√3, -√3, 0, 0}, {√3, 0, 1, √2}, {√3, 0, 1, -√2},
 {√3, 0, -1, √2}, {√3, 0, -1, -√2}, {√3, 0, √2, 1}, {√3, 0, √2, -1},
 {√3, 0, -√2, 1}, {√3, 0, -√2, -1}, {√3, 0, √3, 0}, {√3, 0, -√3, 0},
 {√3, 0, 0, √3}, {√3, 0, 0, -√3}, {-√3, 1, 1, 1}, {-√3, 1, 1, -1},
 {-√3, 1, -1, 1}, {-√3, 1, -1, -1}, {-√3, 1, √2, 0}, {-√3, 1, -√2, 0},
 {-√3, 1, 0, √2}, {-√3, 1, 0, -√2}, {-√3, -1, 1, 1}, {-√3, -1, 1, -1},
 {-√3, -1, -1, 1}, {-√3, -1, -1, -1}, {-√3, -1, √2, 0}, {-√3, -1, -√2, 0},
 {-√3, -1, 0, √2}, {-√3, -1, 0, -√2}, {-√3, √2, 1, 0}, {-√3, √2, -1, 0},
 {-√3, √2, 0, 1}, {-√3, √2, 0, -1}, {-√3, -√2, 1, 0}, {-√3, -√2, -1, 0},
 {-√3, -√2, 0, 1}, {-√3, -√2, 0, -1}, {-√3, √3, 0, 0}, {-√3, -√3, 0, 0},
 {-√3, 0, 1, √2}, {-√3, 0, 1, -√2}, {-√3, 0, -1, √2}, {-√3, 0, -1, -√2},
 {-√3, 0, √2, 1}, {-√3, 0, √2, -1}, {-√3, 0, -√2, 1}, {-√3, 0, -√2, -1},
 {-√3, 0, √3, 0}, {-√3, 0, -√3, 0}, {-√3, 0, 0, √3}, {-√3, 0, 0, -√3}}
```

- **We found 124 null vectors for g. We can think of these as a discretisation of the null cone.**

In[17]:=
```
Length[nullVectors]
```

Out[17]=
```
124
```

- **Verify that all vectors are indeed null vectors**

In[18]:=
```
Union[Table[
   G[nullVectors[[i]]],
   {i, 1, Length[nullVectors]}
  ]]
```

Out[18]=
```
{0}
```

■ Since these vectors are null for g, they are null for gamma

In[19]:=
```
eqs = Table[
   gamma[nullVectors[[i]]],
   {i, 1, Length[nullVectors]}
  ];
```

In[20]:=
```
show[simp[eqs]]
```

Out[20]//MatrixForm=

```
1  :
2  :
3  :
4  :
5  :
6  :
7  :
8  :
9  :
10 :
11 :
12 :
13 :
14 :
15 :
16 :
17 :
18 :
19 :
20 :
21 :
22 :
23 :
24 :
25 :
26 :
27 :
28 :
29 :
30 :
31 :
32 :
33 :
34 :
35 :
36 :
37 :
38 :
39 :
40 :
41 :
42 :
43 :
```

```
44 :
45 :
46 :
47 :
48 :
49 :
50 :
51 :
52 :
53 :
54 :
55 : 9 s1111 + 12 √3 s1112 + 12 √3 s1113 - 12 √3 s1114 + 18 s1122 + 36 s1123 - 36 s1124 + 18 s11
56 : 9 s1111 - 12 √3 s1112 - 12 √3 s1113 + 12 √3 s1114 + 18 s1122 + 36 s1123 - 36 s1124 + 18 s11
57 : 9 s1111 - 12 √3 s1112 + 12 √3 s1113 - 12 √3 s1114 + 18 s1122 - 36 s1123 + 36 s1124 + 18 s11
58 : 9 s1111 + 12 √3 s1112 - 12 √3 s1113 + 12 √3 s1114 + 18 s1122 - 36 s1123 + 36 s1124 + 18 s11
59 : 9 s1111 + 12 √3 s1112 - 12 √3 s1113 - 12 √3 s1114 + 18 s1122 - 36 s1123 - 36 s1124 + 18 s11
60 : 9 s1111 - 12 √3 s1112 + 12 √3 s1113 + 12 √3 s1114 + 18 s1122 - 36 s1123 - 36 s1124 + 18 s11
61 : 9 s1111 - 12 √3 s1112 - 12 √3 s1113 - 12 √3 s1114 + 18 s1122 + 36 s1123 + 36 s1124 + 18 s11
62 : 9 s1111 + 12 √3 s1112 + 12 √3 s1113 + 12 √3 s1114 + 18 s1122 + 36 s1123 + 36 s1124 + 18 s11
```

- **We now have 62 linear equations constraining the coefficients in gamma.**

In[21]:=
```
sol = Solve[toEqs[eqs], Variables[eqs]]
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[21]=
$$\left\{\left\{s1222 \to -s1112,\ s1223 \to -\frac{s1113}{3},\ s1224 \to -\frac{s1114}{3},\ s1233 \to -\frac{s1112}{3},\ s1234 \to 0,\ s1244 \to -\frac{s1112}{3},\ s1333 \to -s1113,\ s1334 \to -\frac{s1114}{3},\ s1344 \to -\frac{s1113}{3},\ s1444 \to -s1114,\ s2222 \to -s1111 - 6\,s1122,\ s2223 \to -3\,s1123,\ s2224 \to -3\,s1124,\ s2233 \to -\frac{s1111}{3} - s1122 - s1133,\ s2234 \to -s1134,\ s2244 \to -\frac{s1111}{3} - s1122 - s1144,\ s2333 \to -3\,s1123,\ s2334 \to -s1124,\ s2344 \to -s1123,\ s2444 \to -3\,s1124,\ s3333 \to -s1111 - 6\,s1133,\ s3334 \to -3\,s1134,\ s3344 \to -\frac{s1111}{3} - s1133 - s1144,\ s3444 \to -3\,s1134,\ s4444 \to -s1111 - 6\,s1144\right\}\right\}$$

In[22]:=
```
Length[sol]
```

Out[22]= 1

In[23]:=
```
vars = {x0, x1, x2, x3};
```

In[24]:=
```
gammaConstrained = FullSimplify[gamma[vars] //. sol[[1]]]
```

Out[24]=
$$\left(x0^2 - x1^2 - x2^2 - x3^2\right)\left(s1111\left(x0^2 + x1^2 + x2^2 + x3^2\right) + 2\left(2\,s1112\,x0\,x1 + 3\,s1122\,x1^2 + x2\,(2\,s1113\,x0 + 6\,s1123\,x1 + 3\,s1133\,x2) + 2\,(s1114\,x0 + 3\,s1124\,x1 + 3\,s1134\,x2)\,x3 + 3\,s1144\,x3^2\right)\right)$$

In[25]:=
$$HH = -\begin{pmatrix} s1111 & 2\,s1112 & 2\,s1113 & 2\,s1114 \\ 2\,s1112 & s1111 + 6\,s1122 & 6\,s1123 & 6\,s1124 \\ 2\,s1113 & 6\,s1123 & s1111 + 6\,s1133 & 6\,s1134 \\ 2\,s1114 & 6\,s1124 & 6\,s1134 & s1111 + 6\,s1144 \end{pmatrix};$$

In[26]:=
```
f1 = G[vars]
f2 = Simplify[vars.HH.vars]
```

Out[26]= $-x0^2 + x1^2 + x2^2 + x3^2$

Out[27]= $-s1111\left(x0^2 + x1^2 + x2^2 + x3^2\right) - 2\left(2\,s1112\,x0\,x1 + 3\,s1122\,x1^2 + 2\,s1113\,x0\,x2 + 6\,s1123\,x1\,x2 + 3\,s1133\,x2^2 + 2\,s1114\,x0\,x3 + 6\,s1124\,x1\,x3 + 6\,s1134\,x2\,x3 + 3\,s1144\,x3^2\right)$

In[28]:=
```
Simplify[gammaConstrained - f1 f2]
```

Out[28]= 0

- **For a similar argument for two Lorentz metrics, see: Richard A. Toupin, Elasticity and electromagnetics, in: Non-Linear Continuum Theories, C.I.M.E. Conference, Bressanone, Italy 1965. C. Truesdell and G. Grioli coordinators. Pp.203-342.**