In[1]:=
```
SetDirectory["~/KappaLib/"];
<< kappaLib-1.1.m
<< helper.m
```

```
KappaLib v1.1

Loading helper.m..
```

- **Metaclass II:**

In[4]:=
```
vars = {x0, x1, x2, x3};

kappa = emMatrixToKappa[{{a1, -b1, 0, 0, 0, 0}, {b1, a1, 0, 0, 0, 0}, {0, 0, a2, 0, 0, -b2}, {0, 1, 0, a1, b1, 0}, {1, 0, 0, -b1, a1, 0}, {0, 0, b2, 0, 0, a2}}];
```

In[7]:=
```
fr = emKappaToFresnel[kappa, vars];
```

In[8]:=
```
FullSimplify[emDet[kappa]]
```

Out[8]=
$$\left(a1^2 + b1^2\right)^2 \left(a2^2 + b2^2\right)$$

- **We assume that Fresnel polynomial factorises:**

In[9]:=
```
A = Table[ToExpression["A" <> ToString[Min[{i, j}]] <> ToString[Max[{i, j}]]],
    {i, 0, 3}, {j, 0, 3}];
B = Table[ToExpression["B" <> ToString[Min[{i, j}]] <> ToString[Max[{i, j}]]],
    {i, 0, 3}, {j, 0, 3}];
A // MatrixForm
B // MatrixForm
factorised = (vars.A.vars) (vars.B.vars);
```

Out[11]//MatrixForm=
$$\begin{pmatrix} A00 & A01 & A02 & A03 \\ A01 & A11 & A12 & A13 \\ A02 & A12 & A22 & A23 \\ A03 & A13 & A23 & A33 \end{pmatrix}$$

Out[12]//MatrixForm=
$$\begin{pmatrix} B00 & B01 & B02 & B03 \\ B01 & B11 & B12 & B13 \\ B02 & B12 & B22 & B23 \\ B03 & B13 & B23 & B33 \end{pmatrix}$$

In[14]:=
```
cons = Union[Flatten[CoefficientList[ fr - factorised, vars]]];
```

In[15]:=
```
cons = simp[cons];
show[cons] // MatrixForm
```

Out[16]//MatrixForm=

$$\begin{pmatrix}
1 & : & 0 \\
2 & : & -A33\,B33 \\
3 & : & -A00\,B00 - b2 \\
4 & : & b1^2\,b2 - A22\,B22 \\
5 & : & -A11\,B11 + b1^2\,b2 \\
6 & : & -2\,(A01\,B00 + A00\,B01) \\
7 & : & -2\,(A02\,B00 + A00\,B02) \\
8 & : & -2\,(A03\,B00 + A00\,B03) \\
9 & : & -2\,(A11\,B01 + A01\,B11) \\
10 & : & -2\,(A12\,B11 + A11\,B12) \\
11 & : & -2\,(A13\,B11 + A11\,B13) \\
12 & : & -2\,(A22\,B02 + A02\,B22) \\
13 & : & -2\,(A22\,B12 + A12\,B22) \\
14 & : & -2\,(A23\,B22 + A22\,B23) \\
15 & : & -2\,(A33\,B03 + A03\,B33) \\
16 & : & -2\,(A33\,B13 + A13\,B33) \\
17 & : & -2\,(A33\,B23 + A23\,B33) \\
18 & : & -A33\,B11 - 4\,A13\,B13 - A11\,B33 \\
19 & : & -A33\,B22 - 4\,A23\,B23 - A22\,B33 \\
20 & : & -A22\,B11 - 4\,A12\,B12 + 2\,b1^2\,b2 - A11\,B22 \\
21 & : & -A33\,B00 - 4\,A03\,B03 + 4\,b1^2\,b2 - A00\,B33 \\
22 & : & -2\,(2\,A12\,B01 + A11\,B02 + A02\,B11 + 2\,A01\,B12) \\
23 & : & -2\,(A13\,B00 + 2\,A03\,B01 + 2\,A01\,B03 + A00\,B13) \\
24 & : & -2\,(A22\,B01 + 2\,A12\,B02 + 2\,A02\,B12 + A01\,B22) \\
25 & : & -2\,(A23\,B00 + 2\,A03\,B02 + 2\,A02\,B03 + A00\,B23) \\
26 & : & -2\,(A23\,B11 + 2\,A13\,B12 + 2\,A12\,B13 + A11\,B23) \\
27 & : & -2\,(2\,A23\,B12 + A22\,B13 + A13\,B22 + 2\,A12\,B23) \\
28 & : & -2\,(A33\,B01 + 2\,A13\,B03 + 2\,A03\,B13 + A01\,B33) \\
29 & : & -2\,(A33\,B02 + 2\,A23\,B03 + 2\,A03\,B23 + A02\,B33) \\
30 & : & -2\,(A33\,B12 + 2\,A23\,B13 + 2\,A13\,B23 + A12\,B33) \\
31 & : & -A11\,B00 - 4\,A01\,B01 + 2\,a1\,b1 - 2\,a2\,b1 - A00\,B11 \\
32 & : & -A22\,B00 - 4\,A02\,B02 - 2\,a1\,b1 + 2\,a2\,b1 - A00\,B22 \\
33 & : & -4\,(A23\,B01 + A13\,B02 + A12\,B03 + A03\,B12 + A02\,B13 + A01\,B23) \\
34 & : & -2\left((a1 - a2)^2 + A12\,B00 + 2\,A02\,B01 + 2\,A01\,B02 - b1^2 + A00\,B12 + b2^2\right) \\
35 & : & -2\left(2\,A13\,B01 + A11\,B03 + A03\,B11 + 2\,A01\,B13 + b1\left((a1 - a2)^2 + b1^2 + b2^2\right)\right) \\
36 & : & -2\left(2\,A23\,B02 + A22\,B03 + b1\left((a1 - a2)^2 + b1^2 + b2^2\right) + A03\,B22 + 2\,A02\,B23\right)
\end{pmatrix}$$

■ Equation (3): By rescaling, we may assume that A00 = 1.

In[17]:=
```
sub = {A00 -> 1, B00 → -b2};
cons = simp[cons //. sub];
show[cons]
```

Out[19]//MatrixForm=

$$\begin{pmatrix}
1 & : & 0 \\
2 & : & -A33\,B33 \\
3 & : & -2\,B01 + 2\,A01\,b2 \\
4 & : & -2\,B02 + 2\,A02\,b2 \\
5 & : & -2\,B03 + 2\,A03\,b2 \\
6 & : & b1^2\,b2 - A22\,B22 \\
7 & : & -A11\,B11 + b1^2\,b2 \\
8 & : & -2\,(A11\,B01 + A01\,B11) \\
9 & : & -2\,(A12\,B11 + A11\,B12) \\
10 & : & -2\,(A13\,B11 + A11\,B13) \\
11 & : & -2\,(A22\,B02 + A02\,B22) \\
12 & : & -2\,(A22\,B12 + A12\,B22) \\
13 & : & -2\,(A23\,B22 + A22\,B23) \\
14 & : & -2\,(A33\,B03 + A03\,B33) \\
15 & : & -2\,(A33\,B13 + A13\,B33) \\
16 & : & -2\,(A33\,B23 + A23\,B33) \\
17 & : & -A33\,B11 - 4\,A13\,B13 - A11\,B33 \\
18 & : & -A33\,B22 - 4\,A23\,B23 - A22\,B33 \\
19 & : & -4\,A03\,B03 + A33\,b2 + 4\,b1^2\,b2 - B33 \\
20 & : & -2\,(2\,A03\,B01 + 2\,A01\,B03 + B13 - A13\,b2) \\
21 & : & -2\,(2\,A03\,B02 + 2\,A02\,B03 - A23\,b2 + B23) \\
22 & : & -A22\,B11 - 4\,A12\,B12 + 2\,b1^2\,b2 - A11\,B22 \\
23 & : & -4\,A01\,B01 + 2\,a1\,b1 - 2\,a2\,b1 - B11 + A11\,b2 \\
24 & : & -4\,A02\,B02 - 2\,a1\,b1 + 2\,a2\,b1 + A22\,b2 - B22 \\
25 & : & -2\,(2\,A12\,B01 + A11\,B02 + A02\,B11 + 2\,A01\,B12) \\
26 & : & -2\,(A22\,B01 + 2\,A12\,B02 + 2\,A02\,B12 + A01\,B22) \\
27 & : & -2\,(A23\,B11 + 2\,A13\,B12 + 2\,A12\,B13 + A11\,B23) \\
28 & : & -2\,(2\,A23\,B12 + A22\,B13 + A13\,B22 + 2\,A12\,B23) \\
29 & : & -2\,(A33\,B01 + 2\,A13\,B03 + 2\,A03\,B13 + A01\,B33) \\
30 & : & -2\,(A33\,B02 + 2\,A23\,B03 + 2\,A03\,B23 + A02\,B33) \\
31 & : & -2\,(A33\,B12 + 2\,A23\,B13 + 2\,A13\,B23 + A12\,B33) \\
32 & : & -4\,(A23\,B01 + A13\,B02 + A12\,B03 + A03\,B12 + A02\,B13 + A01\,B23) \\
33 & : & -2\left((a1 - a2)^2 + 2\,A02\,B01 + 2\,A01\,B02 - b1^2 + B12 - A12\,b2 + b2^2\right) \\
34 & : & -2\left(2\,A13\,B01 + A11\,B03 + A03\,B11 + 2\,A01\,B13 + b1\left((a1 - a2)^2 + b1^2 + b2^2\right)\right) \\
35 & : & -2\left(2\,A23\,B02 + A22\,B03 + b1\left((a1 - a2)^2 + b1^2 + b2^2\right) + A03\,B22 + 2\,A02\,B23\right)
\end{pmatrix}$$

In[20]:=
```
tmp = Join[Take[cons, {3, 5}], Take[cons, {19, 21}], Take[cons, {23, 24}], {cons[[33]]}]
```

Out[20]=
$$\{-2\,B01 + 2\,A01\,b2,\ -2\,B02 + 2\,A02\,b2,\ -2\,B03 + 2\,A03\,b2,\ -4\,A03\,B03 + A33\,b2 + 4\,b1^2\,b2 - B33,$$
$$-2\,(2\,A03\,B01 + 2\,A01\,B03 + B13 - A13\,b2),\ -2\,(2\,A03\,B02 + 2\,A02\,B03 - A23\,b2 + B23),$$
$$-4\,A01\,B01 + 2\,a1\,b1 - 2\,a2\,b1 - B11 + A11\,b2,\ -4\,A02\,B02 - 2\,a1\,b1 + 2\,a2\,b1 + A22\,b2 - B22,$$
$$-2\left((a1 - a2)^2 + 2\,A02\,B01 + 2\,A01\,B02 - b1^2 + B12 - A12\,b2 + b2^2\right)\}$$

In[21]:=
```
Solve[toEqs[tmp], {B01, B02, B03, B33, B13, B23, B11, B22, B12}]
```

Out[21]=
$$\{\{B01 \to A01\,b2,\ B02 \to A02\,b2,\ B03 \to A03\,b2,\ B33 \to -4\,A03^2\,b2 + A33\,b2 + 4\,b1^2\,b2,$$
$$B13 \to -4\,A01\,A03\,b2 + A13\,b2,\ B23 \to -4\,A02\,A03\,b2 + A23\,b2,$$
$$B11 \to 2\,a1\,b1 - 2\,a2\,b1 - 4\,A01^2\,b2 + A11\,b2,\ B22 \to -2\,a1\,b1 + 2\,a2\,b1 - 4\,A02^2\,b2 + A22\,b2,$$
$$B12 \to -a1^2 + 2\,a1\,a2 - a2^2 + b1^2 - 4\,A01\,A02\,b2 + A12\,b2 - b2^2\}\}$$

In[22]:=
```
sub = Join[sub, %[[1]]]
```

Out[22]=
$$\{A00 \to 1,\ B00 \to -b2,\ B01 \to A01\,b2,\ B02 \to A02\,b2,\ B03 \to A03\,b2,$$
$$B33 \to -4\,A03^2\,b2 + A33\,b2 + 4\,b1^2\,b2,\ B13 \to -4\,A01\,A03\,b2 + A13\,b2,\ B23 \to -4\,A02\,A03\,b2 + A23\,b2,$$
$$B11 \to 2\,a1\,b1 - 2\,a2\,b1 - 4\,A01^2\,b2 + A11\,b2,\ B22 \to -2\,a1\,b1 + 2\,a2\,b1 - 4\,A02^2\,b2 + A22\,b2,$$
$$B12 \to -a1^2 + 2\,a1\,a2 - a2^2 + b1^2 - 4\,A01\,A02\,b2 + A12\,b2 - b2^2\}$$

In[23]:=
```
cons = simp[cons //. sub];
show[cons]
```

Out[24]//MatrixForm=

$$\begin{pmatrix}
1 & : & 0 \\
2 & : & 4\,A03\left(2\,A03^2 - A33 - 2\,b1^2\right) \\
3 & : & -A33\left(-4\,A03^2 + A33 + 4\,b1^2\right) \\
4 & : & 4\,A02\left(a1\,b1 - a2\,b1 + 2\,A02^2\,b2 - \right. \\
5 & : & 4\,A01\left(-a1\,b1 + a2\,b1 + 2\,A01^2\,b2 - \right. \\
6 & : & 4\left(-2\,A03\,A13 + A01\left(6\,A03^2 - A33 - 2\right.\right. \\
7 & : & 4\left(-2\,A03\,A23 + A02\left(6\,A03^2 - A33 - 2\right.\right. \\
8 & : & 4\left(2\,A03^2\,A13 + 2\,A01\,A03\,A33 - A13\left(A33\right.\right. \\
9 & : & 4\left(2\,A03^2\,A23 + 2\,A02\,A03\,A33 - A23\left(A33\right.\right. \\
10 & : & b1^2\,b2 + A11\left(-2\,a1\,b1 + 2\,a2\,b1 + 4\,A01^2\right. \\
11 & : & 2\,a1\,A22\,b1 - 2\,a2\,A22\,b1 + \left(4\,A02^2\,A22 - \right. \\
12 & : & 8\,A01\,A03\,A11\,b2 - 4\,A13\left(a1\,b1 - a2\,b1 + \left(-\right.\right. \\
13 & : & 4\,a1\,A23\,b1 + 8\,A02\,A03\,A22\,b2 - 4\,A23\left(a2\,b1 + \right. \\
14 & : & 4\left(-2\,(A02\,A13 + A01\,A23)\,b2 + A03\left((a1 - a2)^2 - b1^2 + b\right.\right. \\
15 & : & 4\left(6\,A01^2\,A02\,b2 - A02\,(a1\,b1 - a2\,b1 + A11\,b2) + A01\left((a1 - \right.\right. \\
16 & : & 4\left(6\,A01\,A02^2\,b2 + A01\,(a1\,b1 - a2\,b1 - A22\,b2) + A02\left((a1 - \right.\right. \\
17 & : & -2\left((a1 - a2)\,(2\,A03 + a1 - a2)\,b1 + b1^3 + 2\left(-6\,A01^2\,A03 + \right.\right. \\
18 & : & -2\left(4\,A02\,A23\,b2 + 2\,A03\left(-a1\,b1 + a2\,b1 - 6\,A02^2\,b2 + A22\,b\right.\right. \\
19 & : & 2\,(a1 - a2)\,A33\,b1 + 2\left(2\,A03^2\,A22 + 8\,A02\,A03\,A23 - 2\,A23^2 + \right. \\
20 & : & 2\left(-a1\,A33\,b1 + a2\,A33\,b1 + \left(2\,A03^2\,A11 + 8\,A01\,A03\,A13 - 2\,A13^2\right.\right. \\
21 & : & 2\left(a1^2\,A11 - 2\,a1\,(A11\,a2 + A12\,b1) + 2\,A12\left(a2\,b1 + 2\,A01^2\,b2\right) + A11\right. \\
22 & : & 2\left(a1^2\,A22 + a2^2\,A22 - 2\,A12\,a2\,b1 + a1\,(-2\,a2\,A22 + 2\,A12\,b1) + 4\,A02^2\,A12\right. \\
23 & : & 2\,A33\left((a1 - a2)^2 - b1^2\right) + 4\left(2\,A03^2\,A12 - 2\,A13\,A23 + 4\,A03\,(A02\,A13 + A01\,A23)\right. \\
24 & : & 4\left(a1^2\,A23 + a2^2\,A23 - A13\,a2\,b1 + a1\,(-2\,a2\,A23 + A13\,b1) + \left(4\,A02\,A03\,A12 + 2\,A02^2\,A13 + 2\,A0\right.\right. \\
25 & : & 4\left(a1^2\,A13 + a2\,A23\,b1 - a1\,(2\,A13\,a2 + A23\,b1) + 2\,A02\,A03\,A11\,b2 + 4\,A01\,A03\,A12\,b2 + 2\,A01^2\right. \\
26 & : & 2\left(2\,a1^2\,A12 + a2\,(-A11 + A22)\,b1 + a1\,(-4\,A12\,a2 + (A11 - A22)\,b1) - 2\,A12^2\,b2 + \left(2\,A02^2\,A11 + 2\right.\right.
\end{pmatrix}$$

In[25]:=
```
Variables[cons]
elimVars = Variables[A]
condVars = Variables[kappa]
```

Out[25]= {A01, A02, A03, a1, A11, A12, A13, a2, A22, A23, A33, b1, b2}

Out[26]= {A00, A01, A02, A03, A11, A12, A13, A22, A23, A33}

Out[27]= {a1, a2, b1, b2}

In[28]:=
```
gb = GroebnerBasis[cons, condVars, elimVars]; // Timing
```

Out[28]= {206.954, Null}

In[29]:=
```
show[simp[gb]]
```

Out[29]//MatrixForm=

$$\begin{pmatrix}
1 & : & b1^5\, b2^2\, \left(b1^2 - b2^2\right)^3 \\
2 & : & (a1 - a2)\, b1^5\, (b1 - b2)\, b2^2\, (b1 + b2) \\
3 & : & (a1 - a2)\, b1^4\, b2^2\, \left((a1 - a2)^2 - b1^2 + b2^2\right) \\
4 & : & (a1 - a2)\, b1^5\, \left((a1 - a2)^2 + (b1 - b2)\, (b1 + b2)\right) \\
5 & : & -b1^5\, b2^2\, \left(b1^4 - 2\, \left(2\, (a1 - a2)^2 + b1^2\right)\, b2^2 + b2^4\right) \\
6 & : & b1^5\, \left((a1 - a2)^2\, b1^2 + b1^4 - \left(5\, (a1 - a2)^2 + 2\, b1^2\right)\, b2^2 + b2^4\right) \\
7 & : & (a1 - a2)\, b1^2\, \left((a1 - a2)^2 - b1^2 + b2^2\right)\, \left((a1 - a2)^2 + b1^2 + b2^2\right) \\
8 & : & -b1^4\, b2^2\, \left(-5\, (a1 - a2)^2\, b1^2 + b1^4 + \left((a1 - a2)^2 - 2\, b1^2\right)\, b2^2 + b2^4\right) \\
9 & : & b1^4\, b2^2\, \left(-5\, b1^6 + 14\, b1^4\, b2^2 - 13\, b1^2\, b2^4 + 4\, b2^4\, \left((a1 - a2)^2 + b2^2\right)\right) \\
10 & : & b1\, \left((a1 - a2)^2 + 2\, (a1 - a2)\, b1 - b1^2 + b2^2\right)\, \left((a1 - a2)^2 + b1^2 + b2^2\right)\, \left(a1^2 + a2^2 + 2\, a2\, b1 - b1^2 - \right. \\
11 & : & b1^2\, \left(-2\, b1^4\, \left((a1 - a2)^2 + b1^2\right) + \left(3\, (a1 - a2)^4 - 8\, (a1 - a2)^2\, b1^2 + 7\, b1^4\right)\, b2^2 + 2\, \left(3\, (a1 - a2)^2\right. \\
12 & : & -b1^4\, \left(-3\, a1^4 + 12\, a1^3\, a2 - 3\, a2^4 + \left(b1^2 - b2^2\right)^2 - 2\, a2^2\, \left(b1^2 + b2^2\right) + 4\, a1\, a2\, \left(3\, a2^2 + b1^2 + b2^2\right) - 2\right.
\end{pmatrix}$$

- **Since b1, b2 have the same signs, equation (1) implies that b1=b2**

In[30]:=
```
show[simp[gb /. b1 → b2]]
```

Out[30]//MatrixForm=

$$\begin{pmatrix}
1 & : & 0 \\
2 & : & (a1 - a2)^3\, b2^5 \\
3 & : & (a1 - a2)^3\, b2^6 \\
4 & : & 4\, (a1 - a2)^2\, b2^8 \\
5 & : & 4\, (a1 - a2)^2\, b2^9 \\
6 & : & -4\, (a1 - a2)^2\, b2^7 \\
7 & : & 4\, (a1 - a2)^2\, b2^{10} \\
8 & : & (a1 - a2)^5\, b2^2 + 2\, (a1 - a2)^3\, b2^4 \\
9 & : & 3\, (a1 - a2)^4\, b2^4 - 4\, (a1 - a2)^2\, b2^6 \\
10 & : & 3\, (a1 - a2)^4\, b2^4 + 4\, (a1 - a2)^2\, b2^6 \\
11 & : & (a1 - a2)^6\, b2 - 2\, (a1 - a2)^4\, b2^3 - 8\, (a1 - a2)^2\, b2^5
\end{pmatrix}$$

In[31]:=
```
show[simp[gb /. {b1 → b2, a1 → a2}]]
```

Out[31]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \end{pmatrix}$$

- **Verify**

In[32]:=
```
AA = {{1, 0, 0, b1}, {0, -b1, 0, 0}, {0, 0, -b1, 0}, {b1, 0, 0, 0}};

BB = {{-1, 0, 0, b1}, {0, -b1, 0, 0}, {0, 0, -b1, 0}, {b1, 0, 0, 0}};

verify = (vars.AA.vars) (vars.(b1 BB).vars);
Simplify[fr - verify /. {a1 → a2, b1 → b2}]
```

Out[35]= 0

- **Check that both metrics have Lorentz signature**

In[36]:=
```
Det[AA]
Det[BB]
```

Out[36]= $-b1^4$

Out[37]= $-b1^4$

■ Extra: Check expressions for inverse matrices

In[38]:=
```
Inverse[AA] // MatrixForm
Inverse[BB] // MatrixForm
```

Out[38]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{b1} \\ 0 & -\frac{1}{b1} & 0 & 0 \\ 0 & 0 & -\frac{1}{b1} & 0 \\ \frac{1}{b1} & 0 & 0 & -\frac{1}{b1^2} \end{pmatrix}$$

Out[39]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{b1} \\ 0 & -\frac{1}{b1} & 0 & 0 \\ 0 & 0 & -\frac{1}{b1} & 0 \\ \frac{1}{b1} & 0 & 0 & \frac{1}{b1^2} \end{pmatrix}$$