In[1]:=
```
SetDirectory["~/KappaLib"];
<< kappaLib-1.2.m
<< helper.m
```

```
Loading KappaLib v1.2

Loading helper.m..
```

- **Define Metaclass V with parameters:**

  alpha_i in R,   beta_i in R\0,   and   beta_i all have same sign.

In[4]:=
$$\texttt{kappa = emMatrixToKappa}\left[\begin{pmatrix} a1 & -b1 & 0 & 0 & 0 & 0 \\ b1 & a1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a2 & 0 & 0 & a3 \\ 0 & 1 & 0 & a1 & b1 & 0 \\ 1 & 0 & 0 & -b1 & a1 & 0 \\ 0 & 0 & a3 & 0 & 0 & a2 \end{pmatrix}\right];$$

- **We may assume that a3 != 0 since otherwise the Fresnel surface contains the 3-plane x0=0**

In[5]:=
```
fr = FullSimplify[emKappaToFresnel[kappa, {x0, x1, x2, x3}]];
```

In[6]:=
```
FullSimplify[fr /. {x0 → 0, a3 → 0}]
```

Out[6]= 0

Write out algebraic equations that kappa satisfies and eliminate variables for A and B

In[7]:=
```
eta = kappa + mu emIdentityKappa[];
LHS = emCompose[eta, eta];
AA = emMatrix["A", 4, Structure → "AntiSymmetric"];
BB = emMatrix["B", 4, Structure → "AntiSymmetric"];
RHS = -lambda emIdentityKappa[] + emBiProduct[rho, AA, BB] + emBiProduct[rho, BB, AA];
```

- **Since rho, A,B are all non-zero, we may scale A and assume that rho = 1**

In[12]:=
```
rho = 1;
```

In[13]:=
```
eqs = simp[Union[Flatten[LHS - RHS]]];
show[eqs]
```

Out[14]//MatrixForm=

1	:	0
2	:	$4\,A24\,B24$
3	:	$-4\,A34\,B34$
4	:	$2\,(b1 - 2\,A12\,B12)$
5	:	$2\,(b1 + 2\,A13\,B13)$
6	:	$2\,(A14\,B13 + A13\,B14)$
7	:	$2\,(A23\,B13 + A13\,B23)$
8	:	$2\,(A24\,B14 + A14\,B24)$
9	:	$2\,(A24\,B23 + A23\,B24)$
10	:	$2\,(A34\,B24 + A24\,B34)$
11	:	$-2\,(A14\,B12 + A12\,B14)$
12	:	$-2\,(A14\,B13 + A13\,B14)$
13	:	$-2\,(A23\,B12 + A12\,B23)$
14	:	$-2\,(A23\,B13 + A13\,B23)$
15	:	$-2\,(A24\,B14 + A14\,B24)$
16	:	$-2\,(A24\,B23 + A23\,B24)$
17	:	$-2\,(A34\,B14 + A14\,B34)$
18	:	$-2\,(A34\,B23 + A23\,B34)$
19	:	$-2\,(A34\,B24 + A24\,B34)$
20	:	$-4\,A14\,B14 + 2\,a3\,(a2 + mu)$
21	:	$-4\,A23\,B23 + 2\,a3\,(a2 + mu)$
22	:	$2\,(a1 - A13\,B12 - A12\,B13 + mu)$
23	:	$-2\,(a1 - A13\,B12 - A12\,B13 + mu)$
24	:	$2\,(A24\,B12 + A12\,B24 + b1\,(a1 + mu))$
25	:	$2\,(A34\,B13 + A13\,B34 + b1\,(a1 + mu))$
26	:	$-2\,(A24\,B12 + A12\,B24 + b1\,(a1 + mu))$
27	:	$-2\,(A34\,B13 + A13\,B34 + b1\,(a1 + mu))$
28	:	$a3^2 - 2\,A23\,B14 - 2\,A14\,B23 + lambda + (a2 + mu)^2$
29	:	$-b1^2 + 2\,A24\,B13 + 2\,A13\,B24 + lambda + (a1 + mu)^2$
30	:	$-b1^2 - 2\,A34\,B12 - 2\,A12\,B34 + lambda + (a1 + mu)^2$

In[15]:=
```
elimVars = Join[Variables[AA], Variables[BB]]
```

Out[15]= {A12, A13, A14, A23, A24, A34, B12, B13, B14, B23, B24, B34}

In[16]:=
```
condVars = Join[Variables[kappa], {lambda, mu}]
```

Out[16]= {a1, a2, a3, b1, lambda, mu}

■ Eliminate variables using a Gröbner basis

In[17]:=
```
gb = GroebnerBasis[eqs, condVars, elimVars]; // Timing
gb = simp[gb]; // Timing
Length[gb]
```

Out[17]= {80.078, Null}

Out[18]= {0.676302, Null}

Out[19]= 31

In[20]:= **show[gb]**

Out[20]//MatrixForm=

```
 1  :  a3 b1^4 (a2 + mu)
 2  :  a3 lambda^2 (a2 + mu)
 3  :  a3 b1 lambda (a2 + mu)
 4  :  a3 b1^4 (a3^2 + lambda)
 5  :  a3 lambda^2 (a3^2 + lambda)
 6  :  lambda (b1^3 - b1 lambda)^3
 7  :  a3 b1^3 (a1 + mu) (a2 + mu)
 8  :  a3 b1 lambda (a3^2 + lambda)
 9  :  a3 lambda (a1 + mu) (a2 + mu)
10  :  a3 b1^3 (a3^2 + lambda) (a1 + mu)
11  :  a3 (a2 + mu) (b1^2 + (a1 + mu)^2)
12  :  b1^4 (a3^2 + lambda + (a2 + mu)^2)
13  :  a3 lambda (a3^2 + lambda) (a1 + mu)
14  :  b1^3 (b1^2 - lambda) lambda (a1 + mu)
15  :  lambda^2 (a3^2 + lambda + (a2 + mu)^2)
16  :  a3 (a3^2 + lambda) (b1^2 + (a1 + mu)^2)
17  :  b1 lambda (a3^2 + lambda + (a2 + mu)^2)
18  :  b1^3 (a1 + mu) (b1^2 - lambda + (a1 + mu)^2)
19  :  b1^3 (a1 + mu) (a3^2 + lambda + (a2 + mu)^2)
20  :  lambda (a1 + mu) (a3^2 + lambda + (a2 + mu)^2)
21  :  b1 lambda (a1 + mu) (-b1^2 + lambda + (a1 + mu)^2)
22  :  (b1^2 + (a1 + mu)^2) (a3^2 + lambda + (a2 + mu)^2)
23  :  b1 (-a3^2 - lambda + (a2 + mu)^2) (-a3^2 + lambda + (a2 + mu
24  :  lambda (-a3^2 - lambda + (a2 + mu)^2) (-a3^2 + lambda + (a2 +
25  :  (a1 + mu) (-a3^2 - lambda + (a2 + mu)^2) (-a3^2 + lambda + (a2 -
26  :  b1 lambda (-5 b1^6 + 14 b1^4 lambda - 13 b1^2 lambda^2 + 4 lambda^2 (lambd
27  :  b1 (b1^2 - lambda) ((b1^2 - lambda)^2 + a1^2 (b1^2 + lambda) + 2 a1 (b1^2 + lambda)
28  :  -b1 lambda ((b1^2 - lambda)^2 + a1^2 (-5 b1^2 + lambda) + 2 a1 (-5 b1^2 + lambda) mu
29  :  lambda (a1^2 - b1^2 + lambda - 2 b1 mu + mu^2 + 2 a1 (-b1 + mu)) (a1^2 - b1^2 + lambda + 2
30  :  b1 (3 a1^4 - (b1^2 - lambda)^2 + 12 a1^3 mu + 2 (b1^2 + lambda) mu^2 + 3 mu^4 + 4 a1 mu (b1^2 + lambda -
31  :  (a1 + mu) (a1^4 - 3 b1^4 + 4 a1^3 mu + 2 b1^2 (lambda - mu^2) + (lambda + mu^2)^2 + 4 a1 mu (-b1^2 + lambd
```

- **By equation (1) we have mu = -a2**

In[21]:=
```
subs = {mu → -a2};
show[simp[gb //. subs]]
```

Out[22]//MatrixForm=
```
1  :  0
2  :  b1^4 (a3^2 + lambda)
3  :  b1 (a3^4 - lambda^2)
4  :  a3 b1^4 (a3^2 + lambda)
5  :  lambda^2 (a3^2 + lambda)
6  :  lambda (a3^4 - lambda^2)
7  :  b1 lambda (a3^2 + lambda)
8  :  a3 lambda^2 (a3^2 + lambda)
9  :  lambda (b1^3 - b1 lambda)^3
10 :  (a1 - a2) (a3^4 - lambda^2)
11 :  a3 b1 lambda (a3^2 + lambda)
12 :  (a1 - a2) b1^3 (a3^2 + lambda)
13 :  (a1 - a2) lambda (a3^2 + lambda)
14 :  (a1 - a2) a3 b1^3 (a3^2 + lambda)
15 :  (a1 - a2) a3 lambda (a3^2 + lambda)
16 :  ((a1 - a2)^2 + b1^2) (a3^2 + lambda)
17 :  (a1 - a2) b1^3 (b1^2 - lambda) lambda
18 :  a3 ((a1 - a2)^2 + b1^2) (a3^2 + lambda)
19 :  (a1 - a2) b1^3 ((a1 - a2)^2 + b1^2 - lambda)
20 :  (a1 - a2) b1 lambda ((a1 - a2)^2 - b1^2 + lambda)
21 :  -b1 lambda (-5 (a1 - a2)^2 b1^2 + b1^4 + ((a1 - a2)^2 - 2 b1^2) lambda
22 :  b1 (b1^2 - lambda) ((a1 - a2)^2 b1^2 + b1^4 + ((a1 - a2)^2 - 2 b1^2) lambd
23 :  b1 lambda (-5 b1^6 + 14 b1^4 lambda - 13 b1^2 lambda^2 + 4 lambda^2 ((a1 - a
24 :  lambda ((a1 - a2)^2 + 2 (a1 - a2) b1 - b1^2 + lambda) (a1^2 + a2^2 + 2 a2 b1 - b1^2 - 2
25 :  b1 (3 a1^4 - 12 a1^3 a2 + 3 a2^4 - (b1^2 - lambda)^2 + 2 a2^2 (b1^2 + lambda) - 4 a1 a2 (3 a2^2 + b1^2 + la
26 :  (a1 - a2) (a1^4 - 4 a1^3 a2 - 3 b1^4 + 2 b1^2 (-a2^2 + lambda) + (a2^2 + lambda)^2 - 4 a1 a2 (a2^2 - b1^2 +
```

- **Equation (2):**

  $b1^4 (a3^2 + lambda) = 0$

contradicts b1 != 0, a3 != 0 and lambda >0. Thus Metaclass V is not possible.