```
In[1]:= SetDirectory["~/KappaLib"];
        << kappaLib-1.2.m
        << helper.m

        Loading KappaLib v1.2

        Loading helper.m..
```

- **Define Metaclass IV with parameters:**

  alpha_i in R, beta_i in R\0, and beta_i all have same sign.

```
In[4]:= kappa = emMatrixToKappa[( a1  0   0   -b1  0   0
                                   0   a2  0   0    -b2 0
                                   0   0   a3  0    0   a4
                                   b1  0   0   a1   0   0
                                   0   b2  0   0    a2  0
                                   0   0   a4  0    0   a3 )];
```

- **We may assume that a4 != 0 since otherwise the Fresnel surface contains the plane x0=x3=0**

```
In[5]:= fr = FullSimplify[emKappaToFresnel[kappa, {x0, x1, x2, x3}]];

In[6]:= fr /. {x0 → 0, x3 → 0, a4 → 0}

Out[6]= 0
```

- **We may assume that a3^2 != a4^2 since otherwise det(kappa)=0**

```
In[7]:= FullSimplify[emDet[kappa]]
```

Out[7]= $(a3 - a4)\,(a3 + a4)\,\left(a1^2 + b1^2\right)\,\left(a2^2 + b2^2\right)$

Write out algebraic equations that kappa satisfies and eliminate variables for A and B

```
In[8]:= eta = kappa + mu emIdentityKappa[];
        LHS = emCompose[eta, eta];
        AA = emMatrix["A", 4, Structure → "AntiSymmetric"];
        BB = emMatrix["B", 4, Structure → "AntiSymmetric"];
        RHS = -lambda emIdentityKappa[] + emBiProduct[rho, AA, BB] + emBiProduct[rho, BB, AA];
```

- **Since rho, A,B are all non-zero, we may scale A and assume that rho = 1**

```
In[13]:= rho = 1;
```

In[14]:=
```
eqs = simp[Union[Flatten[LHS - RHS]]];
show[eqs]
```

Out[15]//MatrixForm=

1	:	0
2	:	2 (A13 B12 + A12 B13)
3	:	2 (A14 B13 + A13 B14)
4	:	2 (A23 B13 + A13 B23)
5	:	2 (A24 B12 + A12 B24)
6	:	2 (A24 B14 + A14 B24)
7	:	2 (A24 B23 + A23 B24)
8	:	2 (A34 B13 + A13 B34)
9	:	2 (A34 B24 + A24 B34)
10	:	- 2 (A13 B12 + A12 B13)
11	:	- 2 (A14 B12 + A12 B14)
12	:	- 2 (A14 B13 + A13 B14)
13	:	- 2 (A23 B12 + A12 B23)
14	:	- 2 (A23 B13 + A13 B23)
15	:	- 2 (A24 B12 + A12 B24)
16	:	- 2 (A24 B14 + A14 B24)
17	:	- 2 (A24 B23 + A23 B24)
18	:	- 2 (A34 B13 + A13 B34)
19	:	- 2 (A34 B14 + A14 B34)
20	:	- 2 (A34 B23 + A23 B34)
21	:	- 2 (A34 B24 + A24 B34)
22	:	4 A13 B13 - 2 b2 (a2 + mu)
23	:	4 A24 B24 + 2 b2 (a2 + mu)
24	:	- 4 A34 B34 - 2 b1 (a1 + mu)
25	:	- 4 A12 B12 + 2 b1 (a1 + mu)
26	:	- 4 A14 B14 + 2 a4 (a3 + mu)
27	:	- 4 A23 B23 + 2 a4 (a3 + mu)
28	:	2 A24 B13 - b2^2 + 2 A13 B24 + lambda + (a2 + mu)^2
29	:	a4^2 - 2 A23 B14 - 2 A14 B23 + lambda + (a3 + mu)^2
30	:	- b1^2 - 2 A34 B12 - 2 A12 B34 + lambda + (a1 + mu)^2

In[16]:=
```
elimVars = Join[Variables[AA], Variables[BB]]
```

Out[16]= {A12, A13, A14, A23, A24, A34, B12, B13, B14, B23, B24, B34}

In[17]:=
```
condVars = Join[Variables[kappa], {lambda, mu}]
```

Out[17]= {a1, a2, a3, a4, b1, b2, lambda, mu}

■ Eliminate variables using a Gröbner basis

In[18]:=
```
gb = GroebnerBasis[eqs, condVars, elimVars]; // Timing
gb = simp[gb]; // Timing
Length[gb]
```

Out[18]= {142.938, Null}

Out[19]= {1.37819, Null}

Out[20]= 45

In[21]:=
```
show[gb]
```

Out[21]//MatrixForm=

1 : $a4\ b1\ b2\ (a1 + mu)\ (a2 + mu)\ (a3 + mu)$

2 : $a4\ b1\ b2\ (a4^2 + lambda)\ (a1 + mu)\ (a2 + m$

3 : $a4\ b1\ b2\ (b2^2 - lambda)\ (a1 + mu)\ (a3 + m$

4 : $a4\ b1\ b2\ (b1^2 - lambda)\ (a2 + mu)\ (a3 + m$

5 : $a4\ b1\ b2\ (b2^2 - lambda)\ (a4^2 + lambda)\ (a1$

6 : $a4\ b1\ b2\ (b1^2 - lambda)\ (a4^2 + lambda)\ (a2$

7 : $a4\ b1\ b2\ (b1^2 - lambda)\ (b2^2 - lambda)\ (a3$

8 : $b1\ b2\ (a1 + mu)\ (a2 + mu)\ (a4^2 + lambda + (a3$

9 : $a4\ b1\ b2\ (b1^2 - lambda)\ (b2^2 - lambda)\ (a4^2 +$

10 : $a4\ b2\ (a2 + mu)\ (a3 + mu)\ (-b1^2 + lambda + (a1$

11 : $a4\ b1\ (a1 + mu)\ (a3 + mu)\ (-b2^2 + lambda + (a2$

12 : $b1\ b2\ (b2^2 - lambda)\ (a1 + mu)\ (a4^2 + lambda + ($

13 : $b1\ b2\ (b1^2 - lambda)\ (a2 + mu)\ (a4^2 + lambda + ($

14 : $a4\ b2\ (a4^2 + lambda)\ (a2 + mu)\ (-b1^2 + lambda +$

15 : $a4\ b2\ (b2^2 - lambda)\ (a3 + mu)\ (-b1^2 + lambda +$

16 : $a4\ b1\ (a4^2 + lambda)\ (a1 + mu)\ (-b2^2 + lambda +$

17 : $a4\ b1\ (b1^2 - lambda)\ (a3 + mu)\ (-b2^2 + lambda +$

18 : $b1\ b2\ (b1^2 - lambda)\ (b2^2 - lambda)\ (a4^2 + lambda$

19 : $a4\ b2\ (b2^2 - lambda)\ (a4^2 + lambda)\ (-b1^2 + lambda$

20 : $a4\ b1\ (b1^2 - lambda)\ (a4^2 + lambda)\ (-b2^2 + lambda$

21 : $b1\ (a1 + mu)\ (lambda + (a3 - a4 + mu)^2)\ (lambda + (a$

22 : $b2\ (a2 + mu)\ (lambda + (a3 - a4 + mu)^2)\ (lambda + (a$

23 : $b2\ (a2 + mu)\ (-b1^2 + lambda + (a1 + mu)^2)\ (a4^2 + lamb$

24 : $b1\ (a1 + mu)\ (-b2^2 + lambda + (a2 + mu)^2)\ (a4^2 + lamb$

25 : $a4\ (a3 + mu)\ (-b1^2 + lambda + (a1 + mu)^2)\ (-b2^2 + lamb$

26 : $b1\ (b1^2 - lambda)\ (lambda + (a3 - a4 + mu)^2)\ (lambda +$

27 : $b2\ (b2^2 - lambda)\ (lambda + (a3 - a4 + mu)^2)\ (lambda +$

28 : $b2\ (b2^2 - lambda)\ (-b1^2 + lambda + (a1 + mu)^2)\ (a4^2 + la$

29 : $b1\ (b1^2 - lambda)\ (-b2^2 + lambda + (a2 + mu)^2)\ (a4^2 + la$

30 : $a4\ (a4^2 + lambda)\ (-b1^2 + lambda + (a1 + mu)^2)\ (-b2^2 + l$

31 : $(-b1^2 + lambda + (a1 + mu)^2)\ (lambda + (a3 - a4 + mu)^2)\ (lam$

32 : $(-b2^2 + lambda + (a2 + mu)^2)\ (lambda + (a3 - a4 + mu)^2)\ (lam$

33 : $(-b1^2 + lambda + (a1 + mu)^2)\ (-b2^2 + lambda + (a2 + mu)^2)\ (a4$

34 : $b2\ (a2 + mu)\ (a1^4 + b1^4 + 4\ a1^3\ mu + 2\ b1^2\ (-lambda + mu^2) + (lambda + mu^2)^2 + 4\ a1\ mu\ (b$

35 : $a4\ (a3 + mu)\ (a1^4 + b1^4 + 4\ a1^3\ mu + 2\ b1^2\ (-lambda + mu^2) + (lambda + mu^2)^2 + 4\ a1\ mu\ (b$

36 : $b1\ (a1 + mu)\ (a2^4 + b2^4 + 4\ a2^3\ mu + 2\ b2^2\ (-lambda + mu^2) + (lambda + mu^2)^2 + 4\ a2\ mu\ (b$

37 : $a4\ (a3 + mu)\ (a2^4 + b2^4 + 4\ a2^3\ mu + 2\ b2^2\ (-lambda + mu^2) + (lambda + mu^2)^2 + 4\ a2\ mu\ (b$

38 : $b2\ (b2^2 - lambda)\ (a1^4 + b1^4 + 4\ a1^3\ mu + 2\ b1^2\ (-lambda + mu^2) + (lambda + mu^2)^2 + 4\ a1\ mu$

39 : $a4\ (a4^2 + lambda)\ (a1^4 + b1^4 + 4\ a1^3\ mu + 2\ b1^2\ (-lambda + mu^2) + (lambda + mu^2)^2 + 4\ a1\ mu$

40 : $b1\ (b1^2 - lambda)\ (a2^4 + b2^4 + 4\ a2^3\ mu + 2\ b2^2\ (-lambda + mu^2) + (lambda + mu^2)^2 + 4\ a2\ mu$

41 : $a4\ (a4^2 + lambda)\ (a2^4 + b2^4 + 4\ a2^3\ mu + 2\ b2^2\ (-lambda + mu^2) + (lambda + mu^2)^2 + 4\ a2\ mu$

42 : $(a4^2 + lambda + (a3 + mu)^2)\ (a1^4 + b1^4 + 4\ a1^3\ mu + 2\ b1^2\ (-lambda + mu^2) + (lambda + mu^2)^2 + 4\ a$

43 : $(a4^2 + lambda + (a3 + mu)^2)\ (a2^4 + b2^4 + 4\ a2^3\ mu + 2\ b2^2\ (-lambda + mu^2) + (lambda + mu^2)^2 + 4\ a$

44 : $(-b2^2 + lambda + (a2 + mu)^2)\ (a1^4 + b1^4 + 4\ a1^3\ mu + 2\ b1^2\ (-lambda + mu^2) + (lambda + mu^2)^2 + 4$

45 : $(-b1^2 + lambda + (a1 + mu)^2)\ (a2^4 + b2^4 + 4\ a2^3\ mu + 2\ b2^2\ (-lambda + mu^2) + (lambda + mu^2)^2 + 4$

- **Since lambda>0, the below equations imply that lambda = b1^2 = b2^2**

In[22]:=
```
show[Take[gb, {26, 27}]]
```

Out[22]//MatrixForm=

$$\begin{pmatrix} 1 & : & b1\left(b1^2 - lambda\right)\left(lambda + (a3 - a4 + mu)^2\right)\left(lambda + (a3 + a4 + mu)^2\right) \\ 2 & : & b2\left(b2^2 - lambda\right)\left(lambda + (a3 - a4 + mu)^2\right)\left(lambda + (a3 + a4 + mu)^2\right) \end{pmatrix}$$

In[23]:=
```
subs = {lambda → b1^2, b2 → b1};
```

In[24]:=
```
tmp = simp[gb //. subs];
show[Take[tmp, {16, 17}]]
```

Out[25]//MatrixForm=

$$\begin{pmatrix} 1 & : & a4\left(a4^2 + b1^2\right)(a1 + mu)^2\left(4\,b1^2 + (a1 + mu)^2\right) \\ 2 & : & a4\left(a4^2 + b1^2\right)(a2 + mu)^2\left(4\,b1^2 + (a2 + mu)^2\right) \end{pmatrix}$$

- **Since a4, b1 != 0, the below equations imply that mu = -a1 and a1 = a2**

In[26]:=
```
subs = Append[subs, mu → - a1]
subs = Append[subs, a2 → a1]
```

Out[26]= $\{lambda \to b1^2,\ b2 \to b1,\ mu \to -a1\}$

Out[27]= $\{lambda \to b1^2,\ b2 \to b1,\ mu \to -a1,\ a2 \to a1\}$

In[28]:=
```
show[simp[gb //. subs]]
```

Out[28]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \end{pmatrix}$$

In[29]:=
```
subs
```

Out[29]= $\{lambda \to b1^2,\ b2 \to b1,\ mu \to -a1,\ a2 \to a1\}$

- **We have shown that**

  a4 != 0,
  a4^2 != a3^2,
  b1 = b2
  a1 = a2.

  Thus the Fresnel surface decomposes into a double light cone.