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Characterisation and representation of non-dissipative electromagnetic medium with two Lorentz null cones

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We study Maxwell's equations on a 4-manifold N with a medium that is non-dissipative and has a linear and pointwise response. In this setting, the medium can be represented by a suitable $\binom{2}{2}$-tensor on the 4-manifold N. Moreover, in each cotangent space on N, the medium defines a *Fresnel surface*. Essentially, the Fresnel surface is a tensorial analogue of the dispersion equation that describes the response of the medium for signals in the geometric optics limit. For example, in an isotropic medium the Fresnel surface is at each point a Lorentz null cone. In a recent paper, Lindell, Favaro, and Bergamin introduced a condition that constrains the polarisation for plane waves. In this paper we show (under suitable assumptions) that a slight strengthening of this condition gives a complete pointwise characterisation of all medium tensors for which the Fresnel surface is the union of two distinct Lorentz null cones. This is, for example, the behaviour in uniaxial media such as calcite. Moreover, using the representation formulas from Lindell *et al.* we obtain a closed form representation formula that pointwise parameterises all medium tensors for which the Fresnel surface is the union of two distinct Lorentz null cones. Both the characterisation and the representation formula are tensorial and do not depend on local coordinates.
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I. INTRODUCTION

We will study the *pre-metric* Maxwell's equations, where Maxwell's equations are written on a 4-manifold N and the electromagnetic medium is described by a suitable antisymmetric $\binom{2}{2}$-tensor κ on N that pointwise is determined by 36 real parameters. In each cotangent space on N, the electromagnetic medium determines a fourth order polynomial surface called the *Fresnel surface* that can be seen as a tensorial analogue of the dispersion equation. The Fresnel surface describes the response of the medium to signals in the geometric optics limit.[16,28,34–36] In this work we will assume that the medium is *skewon-free*. Then there are only 21 free parameters and such a medium is non-dissipative. For example, under suitable assumptions the skewon-free assumption will imply that Poynting's theorem holds. On an orientable manifold one can show that invertible skewon-free $\binom{2}{2}$-tensors are in one-to-one correspondence with *area metrics*. By an area metric, we here mean a $\binom{0}{4}$-tensor on N that defines a symmetric non-degenerate inner product for bivectors. Area metrics appear when studying the propagation of a photon in a vacuum with a first order correction from quantum electrodynamics.[9,38] The Einstein field equations have also been generalised into equations where the unknown field is an area metric.[33] For further examples, see Refs. 34 and 38.

We know that in an isotropic medium like vacuum, the Fresnel surface is a Lorentz null cone at each point in N. That is, Lorentz geometry describes the propagation of light in isotropic media. The converse claim is that isotropic media is the only class of (skewon free and axion free) media where Lorentz geometry describes light propagation. This is a conjecture that was formulated and studied in a number of papers.[15,28,29,31] See also the book[16] by Hehl and Obukhov. The conjecture has been proven in a number of cases: in the absence of magneto-electric effects by Obukhov, Fukui, and Rubilar[28] and for a special class of nonlinear media by Obukhov and Rubilar.[31] Also, on the level of the Fresnel polynomial, Favaro, and Bergamin[11] have shown that the medium is isotropic provided that the Fresnel polynomial is proportional to the square of a Lorentzian quadratic form. In Ref. 6 the conjecture was proven for invertible medium tensors with no skewon component and no axion component. Below this result is stated as equivalence (i) $\Leftrightarrow$ (iii) in Theorem 3.2. For additional results and discussions see also Refs. 7, 16, 17, 20, 32, and 35.

Since the Fresnel surface is a 4th order polynomial surface, the Fresnel surface can also decompose into the union of two distinct Lorentz null cones. For example, this is the case in *uniaxial media* like calcite (Ref. 2, Sec. 15.3). In such a medium, the propagation properties of the medium do not only depend on the direction, but also on the polarisation of the wave. In a uniaxial medium, there are two eigenpolarisations and one null cone for each polarisation. In consequence, there is one Fermat's principle for each polarisation.[34] This is the source for the physical phenomenon of double refraction.

We know that for a *uniaxial medium* wave propagation is determined by two distinct null cones. A natural next task is to understand the structure of all medium tensors with this property. This is the main result in Ref. 7, which gives the complete local description of all non-dissipative medium tensors for which the Fresnel surface is two Lorentz null cones (up to suitable assumptions). The importance of this result is that it shows that there are three and only three medium classes with this behaviour. Moreover, the theorem gives explicit expressions for each medium class in local coordinates. The first medium class is a slight generalisation of uniaxial media. The second and third classes seem to be new classes of media. The second class has the peculiar property that there can be three different signal speeds in one spatial direction. In the below, this result is summarised in Theorem 3.4.

The main contribution of this paper is Theorem 5.1. Under suitable assumptions, this theorem gives a pointwise characterisation (condition (ii) in Theorem 5.1) of all non-dissipative medium tensors for which the Fresnel surface is two distinct Lorentz null cones. In a suitable limit, this condition also reduces to the *closure condition* $\kappa^2 = -\lambda \operatorname{Id}$ for a $\lambda > 0$ that characterises a medium with a single Lorentz null cone.[16] Moreover, in Theorem 5.1 we give a tensorial representation formula (Eq. (62)) that parameterises all non-dissipative medium tensors with two distinct Lorentz null cones. Let us emphasise that the characterisation and representation formulas in Theorem 5.1 are tensorial and do not depend on local coordinates. This is the main difference between Theorem 5.1 and the result in[7] mentioned in the above. In[7] the representation formulas relied heavily on coordinate expressions given by the normal form theorem for non-dissipative media by Schuller, Witte, and Wohlfarth.[38] In the proof of Theorem 5.1, we will also use this normal form theorem for computations. However, the end result will be tensorial and independent of coordinates.

The background and motivation for Theorem 5.1 comes from a recent paper by Lindell, Favaro, and Bergamin.[24] In Sec. IV we will briefly summarise some of the results from Ref. 24. In this paper, the authors introduces a second order polynomial condition on the medium tensor (Eq. (53) in the below). Equation (53) is derived from a constraint on polarisation of plane waves, and in Ref. 24 it is shown that whenever condition (53) is satisfied (plus some additional assumptions), the Fresnel surface always factorises into two second order surfaces. In Sec. IV C we will further motivate that Eq. (53) is in fact a general factorisability condition for the Fresnel surface. At first this might seem unexpected since Eq. (53) was initially derived from a constraint on polarisation, yet it is able to constrain the behaviour of signal speed. However, the explanation is that for electromagnetic waves, polarisation, and signal speed are not independent properties but tied together. In Theorem 5.1, condition (ii) is a slight strengthening of Eq. (53). Also, representation formula (62) in Theorem 5.1 is adapted from Ref. 24 and constitute a subclass of generalised Q-medium introduced by Lindell and Wallén in Ref. 25. A further technical discussion on Theorem 5.1 is given in the end of Sec. V.

Some of the computations in the paper rely on computer algebra. Mathematica notebooks for these computations can be found on the author's homepage.

II. PRELIMINARIES

By a *manifold* N we mean a second countable topological Hausdorff space that is locally homeomorphic to $\mathbb{R}^n$ with C^∞-smooth transition maps. All objects are assumed to be smooth where defined. Let TN and T^*N be the tangent and cotangent bundles, respectively. For $k \geq 1$, let $\Omega^k(N)$ be antisymmetric tensor fields with k lower indices (that is, k-forms). Similarly, let $\Omega_k(N)$ be antisymmetric tensor fields with k upper indices. Moreover, let $\Omega^2{}_2(N) = \Omega^2(N) \otimes \Omega_2(N)$. Let also $C^\infty(N)$ be the set of scalar functions (that is, $\binom{0}{0}$-tensors). The Einstein summing convention is used throughout. When writing tensors in local coordinates we assume that the components satisfy the same symmetries as the tensor.

A. Twisted tensors

To formulate Maxwell's equations, we will also need *twisted tensors* (Ref. 16, Sec. A.2.6) in addition to usual tensors. We will denoted these by a tilde over the tensor space. For example, by $\widetilde{\Omega}^2(N)$ we denote the space of twisted 2-forms. If $G \in \widetilde{\Omega}^2(N)$ then in each coordinate chart (U, x^i), G is determined by a usual 2-form $G|_U \in \Omega^2(U)$ and on overlapping charts (U, x^i) and $(\widetilde{U}, \widetilde{x}^i)$, forms $G|_U$ and $G|_{\widetilde{U}}$ satisfy the transformation rule,

$$G|_{\widetilde{U}} = \operatorname{sgn}\det\left(\frac{\partial x^a}{\partial \widetilde{x}^b}\right) G|_U, \tag{1}$$

where $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ is the *sign function*, $\operatorname{sgn} x = x/|x|$ for $x \neq 0$ and $\operatorname{sgn} x = 0$ for $x = 0$. If locally

$$G|_U = \frac{1}{2} G_{ij}\, dx^i \wedge dx^j, \quad G|_{\widetilde{U}} = \frac{1}{2} \widetilde{G}_{ij}\, d\widetilde{x}^i \wedge d\widetilde{x}^j, \tag{2}$$

then Eq. (1) implies that components G_{ij} and $\widetilde{G}_{ij}$ transform as

$$\widetilde{G}_{ij} = \operatorname{sgn}\det\left(\frac{\partial x^a}{\partial \widetilde{x}^b}\right) G_{rs} \frac{\partial x^r}{\partial \widetilde{x}^i} \frac{\partial x^s}{\partial \widetilde{x}^j}. \tag{3}$$

When the chart is clear from context, we will simply write $G = \frac{1}{2} G_{ij}\, dx^i \wedge dx^j$. Similarly, if $\kappa \in \widetilde{\Omega}^2{}_2(N)$ then in each chart κ is represented by a $\kappa|_U \in \Omega^2{}_2(U)$ and locally

$$\kappa = \frac{1}{8} \kappa^{ij}_{rs}\, dx^r \wedge dx^s \otimes \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \tag{4}$$

for suitable components κ^{ij}_{rs}. Moreover, if κ^{ij}_{rs} and $\widetilde{\kappa}^{ij}_{rs}$ are components for κ in overlapping charts (U, x^i) and $(\widetilde{U}, \widetilde{x}^i)$ then we obtain the transformation rule

$$\widetilde{\kappa}^{ij}_{rs} = \operatorname{sgn}\det\left(\frac{\partial x^a}{\partial \widetilde{x}^b}\right) \kappa^{pq}_{uv} \frac{\partial x^u}{\partial \widetilde{x}^r} \frac{\partial x^v}{\partial \widetilde{x}^s} \frac{\partial \widetilde{x}^i}{\partial x^p} \frac{\partial \widetilde{x}^j}{\partial x^q}. \tag{5}$$

Compositions involving twisted tensors are computed in the natural way by composing local tensors. For example, if $\kappa, \eta \in \widetilde{\Omega}^2{}_2(N)$ their composition defines an element $\kappa \circ \eta \in \Omega^2{}_2(N)$ and if κ, η and $\kappa \circ \eta$ are written as in Eq. (4) then

$$(\kappa \circ \eta)^{ij}_{rs} = \frac{1}{2} \kappa^{ab}_{rs} \eta^{ij}_{ab}. \tag{6}$$

If M is orientable and oriented, then twisted tensors coincide with their normal (or untwisted) counterparts. For example, in this case Eq. (5) implies that $\widetilde{\Omega}^2{}_2(N) = \Omega^2{}_2(N)$.

B. Tensor densities

In addition to tensors and twisted tensors, we will need tensor densities and twisted tensor densities. A $\binom{p}{q}$-*tensor density of weight* $w \in \mathbb{Z}$ on a manifold N is determined by components $T^{a_1 \ldots a_p}_{b_1 \cdots b_q}$ in each chart (U, x^i), and on overlapping charts (U, x^i) and $(\widetilde{U}, \widetilde{x}^i)$ we have the transformation rule,[39]

$$\widetilde{T}^{a_1\dots a_p}_{b_1\cdots b_q} = \left(\det\left(\frac{\partial x^i}{\partial \widetilde{x}^j}\right)\right)^w T^{r_1\dots r_p}_{s_1\cdots s_q}\frac{\partial x^{s_1}}{\partial \widetilde{x}^{b_1}}\cdots\frac{\partial x^{s_q}}{\partial \widetilde{x}^{b_q}}\frac{\partial \widetilde{x}^{a_1}}{\partial x^{r_1}}\cdots\frac{\partial \widetilde{x}^{a_p}}{\partial x^{r_p}}.$$

A *twisted* $\binom{p}{q}$*-tensor density of weight* $w \in \mathbb{Z}$ on N is defined in the same way, but with an additional $\operatorname{sgn}\det\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right)$ factor in the transformation rule as in Eqs. (3) and (5).

The *Levi-Civita permutation symbols* are denoted by ε_{ijkl} and ε^{ijkl}. Even if these coincide as combinatorial functions so that $\varepsilon_{ijkl} = \varepsilon^{ijkl}$, they are also different as they globally define different objects on a manifold. Namely, if ε_{ijkl}, ε^{ijkl}, and $\widetilde{\varepsilon}_{ijkl}, \widetilde{\varepsilon}^{ijkl}$ are defined on overlapping coordinate charts (U, x^i) and $(\widetilde{U}, \widetilde{x}^i)$, respectively, then

$$\widetilde{\varepsilon}_{abcd} = \det\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right)\varepsilon_{pqrs}\frac{\partial x^p}{\partial \widetilde{x}^a}\frac{\partial x^q}{\partial \widetilde{x}^b}\frac{\partial x^r}{\partial \widetilde{x}^c}\frac{\partial x^s}{\partial \widetilde{x}^d}, \tag{7}$$

$$\widetilde{\varepsilon}^{abcd} = \det\left(\frac{\partial x^i}{\partial \widetilde{x}^j}\right)\varepsilon^{pqrs}\frac{\partial \widetilde{x}^a}{\partial x^p}\frac{\partial \widetilde{x}^b}{\partial x^q}\frac{\partial \widetilde{x}^c}{\partial x^r}\frac{\partial \widetilde{x}^d}{\partial x^s}. \tag{8}$$

That is, ε_{ijkl} defines a $\binom{0}{4}$-tensor density of weight -1 on N and ε^{ijkl} defines a $\binom{4}{0}$-tensor density of weight 1. For future reference, let us note that

$$\varepsilon^{rsab}\varepsilon_{rsij} = 4\delta^a_{[i}\delta^b_{j]}, \quad \varepsilon^{rabc}\varepsilon_{rijk} = 3!\delta^a_{[i}\delta^b_j\delta^c_{k]}, \tag{9}$$

where δ^i_j is the *Kronecker delta symbol* and brackets $[i_1\dots i_p]$ indicate that indices $i_1, \dots, i_p$ are antisymmetrised with scaling $1/p!$.

C. Maxwell's equations on a 4-manifold

On a 4-manifold N, the *premetric Maxwell's equations* read

$$dF = 0, \tag{10}$$

$$dG = J, \tag{11}$$

$$G = \kappa(F), \tag{12}$$

where d is the exterior derivative, $F \in \Omega^2(N)$, $G \in \widetilde{\Omega}^2(N)$, $J \in \widetilde{\Omega}^3(N)$, and $\kappa \in \widetilde{\Omega}^2{}_2(N)$. Here, F, G, are called the *electromagnetic field variables*, J describes the electromagnetic sources, tensor κ models the electromagnetic medium and Eq. (12) is known as the *constitutive equation*. In local coordinates, Eqs. (10)–(12) reduce to the usual Maxwell's equations. For a systematic treatment, see Refs. 16 and 36.

If locally $F = \frac{1}{2}F_{ij}dx^i \wedge dx^j$, $G = \frac{1}{2}G_{ij}dx^i \wedge dx^j$, and κ is written as in Eq. (4) then constitutive equation (12) is equivalent with

$$G_{ij} = \frac{1}{2}\kappa^{ab}_{ij}F_{ab}. \tag{13}$$

Thus Eq. (12) models an electromagnetic medium with a linear and pointwise response.

Suppose $\kappa \in \widetilde{\Omega}^2{}_2(N)$ and suppose (U, x^i) is a chart. Then the local representation of κ in Eq. (4) defines a pointwise linear map $\Omega^2(U) \to \Omega^2(U)$. In U we can therefore represent κ by a smoothly varying 6×6 matrix. To do this, let O be the ordered set of index pairs $\{01, 02, 03, 23, 31, 12\}$, and if $J \in O$, let $dx^J = dx^{J_1} \wedge dx^{J_2}$, where J_1 and J_2 are the individual indices for J. Say, if $J = 31$ then $dx^J = dx^3 \wedge dx^1$. Then a basis for $\Omega^2(U)$ is given by $\{dx^J : J \in O\}$, that is,

$$\{dx^0 \wedge dx^1, dx^0 \wedge dx^2, dx^0 \wedge dx^3, dx^2 \wedge dx^3, dx^3 \wedge dx^1, dx^1 \wedge dx^2\}. \tag{14}$$

This choice of basis follows Ref. 16, Sec. A.1.10. By Eq. (4) it follows that

$$\kappa(dx^J) = \sum_{I \in O} \kappa_I^J dx^I, \quad J \in O, \tag{15}$$

where $\kappa_I^J = \kappa_{I_1 I_2}^{J_1 J_2}$. Let b be the natural bijection b: $O \to \{1, \dots, 6\}$. Then we identify coefficients $\{\kappa_I^J : I, J \in O\}$ for κ with the smoothly varying 6×6 matrix $P = (\kappa_I^J)_{IJ}$ defined as $\kappa_I^J = P_{b(I)b(J)}$ for $I, J \in O$.

Suppose $P = (\kappa_I^J)_{IJ}$ and $\widetilde{P} = (\widetilde{\kappa}_I^J)_{IJ}$ are smoothly varying 6×6 matrices that represent tensor κ in overlapping charts (U, x^i) and $(\widetilde{U}, \widetilde{x}^i)$. Then Eq. (5) is equivalent with

$$\widetilde{\kappa}_I^J = \operatorname{sgn}\det\left(\frac{\partial x^i}{\partial \widetilde{x}^j}\right) \sum_{K,L \in O} \frac{\partial x^K}{\partial \widetilde{x}^I} \kappa_K^L \frac{\partial \widetilde{x}^J}{\partial x^L}, \quad I, J \in O,$$

where

$$\frac{\partial x^J}{\partial \widetilde{x}^I} = \frac{\partial x^{J_1}}{\partial \widetilde{x}^{I_1}} \frac{\partial x^{J_2}}{\partial \widetilde{x}^{I_2}} - \frac{\partial x^{J_2}}{\partial \widetilde{x}^{I_1}} \frac{\partial x^{J_1}}{\partial \widetilde{x}^{I_2}}, \quad I, J \in O, \tag{16}$$

and $\frac{\partial \widetilde{x}^J}{\partial x^I}$ is defined similarly by exchanging x and $\widetilde{x}$. For matrices $T = (\frac{\partial x^J}{\partial \widetilde{x}^I})_{IJ}$ and $S = (\frac{\partial \widetilde{x}^J}{\partial x^I})_{IJ}$, we have $T = S^{-1}$, whence Eq. (5) is further equivalent with the matrix equation,

$$\widetilde{P} = \operatorname{sgn}\det\left(\frac{\partial x^i}{\partial \widetilde{x}^j}\right) \; TPT^{-1}. \tag{17}$$

In a chart (U, x^i), we define $\operatorname{trace}\kappa : U \to \mathbb{R}$ and $\det\kappa : U \to \mathbb{R}$ as the trace and determinant of the pointwise linear map $\Omega^2(U) \to \Omega^2(U)$. When P is as above it follows that $\operatorname{trace}\kappa = \operatorname{trace} P$ and $\det\kappa = \det P$. When these definitions are extended into each chart on N, Eq. (17) shows that $\operatorname{trace}\kappa \in \widetilde{C}^\infty(N)$ and $\det\kappa \in C^\infty(N)$. Moreover, if κ is written as in Eq. (4), then

$$\operatorname{trace}\kappa = \frac{1}{2}\kappa_{ij}^{ij}.$$

At a point $p \in N$ we say that κ is *invertible* if $(\det\kappa)|_p \neq 0$. If Id is the identity tensor $\mathrm{Id} \in \Omega^2{}_2(N)$, then writing Id as in Eq. (4) gives $\mathrm{Id}_{rs}^{ij} = \delta_r^i\delta_s^j - \delta_s^i\delta_r^j$. For $f \in \widetilde{C}^\infty(N)$ it follows that $\operatorname{trace} f\,\mathrm{Id} = 6f$.

D. Decomposition of electromagnetic medium

At each point of a 4-manifold N, an element of $\widetilde{\Omega}^2{}_2(N)$ depends on 36 parameters. Pointwise, such $\binom{2}{2}$-tensors canonically decompose into three linear subspaces. The motivation for this decomposition is that different components in the decomposition enter in different parts of electromagnetics. See Ref. 16, Sec. D.1.3.

Proposition 2.1: Let N be a 4*-manifold, and let*

$$\begin{aligned} Z &= \{\kappa \in \widetilde{\Omega}^2{}_2(N) : u \wedge \kappa(v) = \kappa(u) \wedge v \textit{ for all } u, v \in \Omega^2(N), \\ &\qquad \operatorname{trace}\kappa = 0\}, \\ W &= \{\kappa \in \widetilde{\Omega}^2{}_2(N) : u \wedge \kappa(v) = -\kappa(u) \wedge v \textit{ for all } u, v \in \Omega^2(N)\}, \\ U &= \{f\,\mathrm{Id} \in \widetilde{\Omega}^2{}_2(N) : f \in \widetilde{C}^\infty(N)\}. \end{aligned}$$

Then

$$\widetilde{\Omega}^2{}_2(N) = Z \oplus W \oplus U, \tag{18}$$

and pointwise, $\dim Z = 20$, $\dim W = 15$ *and* $\dim U = 1$.

If we write a $\kappa \in \widetilde{\Omega}^2{}_2(N)$ as $\kappa = {}^{(1)}\kappa + {}^{(2)}\kappa + {}^{(3)}\kappa$ with ${}^{(1)}\kappa \in Z$, ${}^{(2)}\kappa \in W$, ${}^{(3)}\kappa \in U$, then we say that ${}^{(1)}\kappa$ is the *principal part*, ${}^{(2)}\kappa$ is the *skewon part* and ${}^{(3)}\kappa$ is the *axion part* of κ.[16] For a proof of Proposition 2.1 as stated above, see Ref. 6, and for further discussions, see Refs. 10, 16, and 36.

In $\widetilde{\Omega}^2{}_2(N)$ there is a canonical isomorphism $\widetilde{\Omega}^2{}_2(N) \to \widetilde{\Omega}^2{}_2(N)$[10] related to the *Poincaré isomorphism*.[13] Let us first give a local definition. If $\kappa \in \widetilde{\Omega}^2{}_2(N)$ on a 4-manifold N, we define $\overline{\kappa}$ as the element $\overline{\kappa} \in \widetilde{\Omega}^2{}_2(N)$ defined as

$$\overline{\kappa}^{ij}_{rs} = \frac{1}{4}\varepsilon_{rsab}\kappa^{ab}_{cd}\varepsilon^{cdij} \tag{19}$$

when κ and $\overline{\kappa}$ are written as in Eq. (4). Equations (7) and (8) imply that this assignment defines an element $\overline{\kappa} \in \widetilde{\Omega}^2{}_2(N)$. For $\kappa \in \Omega^2{}_2(N)$ we define $\overline{\kappa}$ in the same way and we also have a canonical isomorphism $\Omega^2{}_2(N) \to \Omega^2{}_2(N)$.

The next proposition collects results for $\overline{\kappa}$. In particular, part (i) states that $\overline{\kappa}$ can be interpreted as a formal adjoint of κ with respect to the wedge product for 2-forms. In consequence, the isomorphism in $\widetilde{\Omega}^2{}_2(N)$ is closely related to the decomposition in Proposition 2.1. For example, a tensor $\kappa \in \widetilde{\Omega}^2{}_2(N)$ has only a principal part if and only if $\kappa = \overline{\kappa}$ and $\operatorname{trace}\kappa = 0$. For a further discussion, see Ref. 10.

Proposition 2.2: Suppose N is a 4*-manifold and $\kappa \in \widetilde{\Omega}^2{}_2(N)$.*

(i) $\overline{\kappa}$ is the unique $\overline{\kappa} \in \widetilde{\Omega}^2_2(N)$ such that

$$\kappa(u) \wedge v = u \wedge \overline{\kappa}(v) \quad \textit{for all } u, v \in \Omega^2(N). \tag{20}$$

(ii) $\overline{f\,\mathrm{Id}} = f\,\mathrm{Id}$ *for all* $f \in \widetilde{C}^\infty(N)$.
(iii) $\overline{\overline{\kappa}} = \kappa$ *and if* $\eta \in \widetilde{\Omega}^2{}_2(N)$, *then* $\overline{\kappa \circ \eta} = \overline{\eta} \circ \overline{\kappa}$.
(iv) $\operatorname{trace}\overline{\kappa} = \operatorname{trace}\kappa$.
(v) If $u \wedge \kappa(u) = 0$ *holds for all* $u \in \Omega^2(N)$ *then* $\kappa + \overline{\kappa} = 0$.

Proof: Part (i) follows by writing out both sides in Eq. (20) in coordinates. Parts (ii) and (iii) follow by part (i). Part (iv) is a direct computation. For part (v) we have

$$u \wedge (\kappa + \overline{\kappa})(v) = \frac{1}{2}\left((u+v) \wedge \kappa(u+v) - (u-v) \wedge \kappa(u-v)\right)$$

for all $u, v \in \Omega^2(N)$, and the claim follows since the right hand side vanishes. □

If ρ is a twisted scalar tensor density of weight 1 on a 4-manifold N and $A, B \in \Omega_2(N)$ then we define $\rho\,\widehat{A} \otimes B$ as the twisted tensor in $\widetilde{\Omega}^2{}_2(N)$ defined as follows. If locally $A = \frac{1}{2}A^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$ and $B = \frac{1}{2}B^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$ then

$$(\rho\,\widehat{A} \otimes B)^{ij}_{rs} = \rho\varepsilon_{rsab}A^{ab}B^{ij}, \tag{21}$$

when $\rho\,\widehat{A} \otimes B$ is written as in Eq. (4). That $\rho\,\widehat{A} \otimes B$ transforms as an element in $\widetilde{\Omega}^2{}_2(N)$ follows by Eq. (7). Similarly when ρ is an untwisted scalar density we define $\rho\,\widehat{A} \otimes B \in \Omega^2{}_2(N)$ by Eq. (21). For both twisted and untwisted ρ we have identities

$$\overline{\rho\,\widehat{A} \otimes B} = \rho\,\widehat{B} \otimes A, \tag{22}$$

$$(\rho\,\widehat{A} \otimes B) \circ \kappa = \rho\,\widehat{A} \otimes (B\kappa), \tag{23}$$

$$\overline{\kappa} \circ (\rho\,\widehat{A} \otimes B) = \rho\,\widehat{(A\kappa)} \otimes B, \tag{24}$$

$$(\rho\,\widehat{A} \otimes B) \circ (\rho\,\widehat{B} \otimes A) = \operatorname{trace}(\rho\,\widehat{B} \otimes B)\,(\rho\,\widehat{A} \otimes A). \tag{25}$$

In Sec. IV B and in the proof of Theorem 5.1 we will need the following lemma.

Lemma 2.3: Suppose N is a 4-manifold and $\kappa \in \widetilde{\Omega}^2{}_2(N)$ is defined as

$$\kappa = \rho \left(\widehat{A} \otimes B + \widehat{B} \otimes A\right) + f \,\mathrm{Id}, \tag{26}$$

where ρ is a scalar tensor density of weight 1, $A, B \in \Omega_2(N)$, and $f \in \widetilde{C}^\infty(N)$. Then $\kappa|_p = 0$ at a point $p \in N$ implies that $f|_p = 0$ and $\rho|_p = 0$ or $A|_p = 0$ or $B|_p = 0$.

If κ is written as in Eq. (4) and A, B are written as above, then Eq. (26) states that

$$\kappa^{ij}_{rs} = \rho \varepsilon_{rsab} \left(A^{ab} B^{ij} + A^{ij} B^{ab}\right) + f \,\mathrm{Id}^{ij}_{rs} .$$

Proof: By restricting the analysis to p and introducing notation $A^I = A^{I_1 I_2}$ and $B^I = B^{I_1 I_2}$, we obtain

$$2\rho(A^I B^J + A^J B^I) + f \varepsilon^{IJ} = 0 \quad \text{for all } I, J \in O. \tag{27}$$

Setting $I = J$ and summing implies that $\sum_{I \in O} \rho A^I B^I = 0$. Multiplying each Eq. in (27) by $A^I B^J$ and ε^{IJ} and summing I, J yields two scalar equations. Eliminating f from these equations gives

$$\rho \left(\left(\sum_{I \in O} (A^I)^2\right) \left(\sum_{I \in O} (B^I)^2\right) + \frac{1}{3} \left(\sum_{I, J \in O} \varepsilon^{IJ} A^I B^J\right)^2 \right) = 0,$$

and the claim follows. □

E. The Fresnel surface

Let $\kappa \in \widetilde{\Omega}^2{}_2(N)$ on a 4-manifold N. If κ is locally given by Eq. (4) in coordinates $\{x^i\}$, let

$$\mathscr{G}_0^{ijkl} = \frac{1}{48} \kappa^{a_1 a_2}_{b_1 b_2} \kappa^{a_3 i}_{b_3 b_4} \kappa^{a_4 j}_{b_5 b_6} \varepsilon^{b_1 b_2 b_5 k} \varepsilon^{b_3 b_4 b_6 l} \varepsilon_{a_1 a_2 a_3 a_4}. \tag{28}$$

If $\{\widetilde{x}^i\}$ are overlapping coordinates, then Eqs. (5), (7), and (8) imply that components $\mathscr{G}_0^{ijkl}$ satisfy the transformation rule

$$\widetilde{\mathscr{G}}_0^{ijkl} = \left|\det\left(\frac{\partial x^r}{\partial \widetilde{x}^s}\right)\right| \mathscr{G}_0^{abcd} \frac{\partial \widetilde{x}^i}{\partial x^a} \frac{\partial \widetilde{x}^j}{\partial x^b} \frac{\partial \widetilde{x}^k}{\partial x^c} \frac{\partial \widetilde{x}^l}{\partial x^d}. \tag{29}$$

Thus components $\mathscr{G}_0^{ijkl}$ define a twisted $\binom{4}{0}$-tensor density $\mathscr{G}_0$ on N of weight 1. The *Tamm-Rubilar tensor density*[16,36] is the totally symmetric part of $\mathscr{G}_0$ and we denote this twisted tensor density by $\mathscr{G}$. In coordinates, $\mathscr{G}^{ijkl} = \mathscr{G}_0^{(ijkl)}$, where parentheses indicate that indices $ijkl$ are symmetrised with scaling 1/4!. If locally $\xi = \xi_i dx^i$ it follows that $\mathscr{G}^{ijkl} \xi_i \xi_j \xi_k \xi_l = \mathscr{G}_0^{ijkl} \xi_i \xi_j \xi_k \xi_l$, and we call $\mathscr{G}^{ijkl} \xi_i \xi_j \xi_k \xi_l$ the *Fresnel polynomial*.

The *Fresnel surface* at a point $p \in N$ is defined as

$$F_p(\kappa) = \{\xi \in T^*_p(N) : \mathscr{G}^{ijkl} \xi_i \xi_j \xi_k \xi_l = 0\}. \tag{30}$$

By Eq. (29), the definition of $F_p(\kappa)$ does not depend on local coordinates. Let $F(\kappa) = \coprod_{p \in N} F_p(\kappa)$ be the disjoint union of all Fresnel surfaces.

The Fresnel surface $F(\kappa)$ is a fundamental object when studying wave propagation in Maxwell's equations. Essentially, equation $\mathscr{G}^{ijkl} \xi_i \xi_j \xi_k \xi_l = 0$ in Eq. (30) is a tensorial analogue to the dispersion equation that describes wave propagation in the geometric optics limit. Thus $F(\kappa)$ constrains possible wave speed(s) as a function of direction. In general the Fresnel surface $F_p(\kappa)$ is a fourth order polynomial surface in $T^*_p(N)$, so it can have multiple sheets and singular points.[30]

There are various ways to derive the Fresnel surface; by studying a propagating weak singularity,[16,28,36] using a geometric optics,[6,18] or as the characteristic polynomial of the full Maxwell's equations.[38] The tensorial description of the Fresnel surface is due to Obukhov, Fukui, and Rubilar.[28]

III. RESULTS FOR SKEWON-FREE MEDIUM

In this section we collect a number of results for twisted skewon-free tensors that we will need in the proof of Theorem 5.1.

A. The normal form theorem by Schuller *et al.*

The normal form theorem for skewon-free medium by Schuller, Witte, and Wohlfarth[38] shows that there exists 23 simple normal form matrices such that any skewon-free medium tensor can pointwise be transformed into one of these normal forms by a coordinate transformation plus, possibly, a conjugation by a Hodge operator. Next, we formulate a slightly simplified version of this result that is sufficiently general for the proof of Theorem 5.1. Let us note that the original theorem in Ref. 38 is formulated for *area metrics*. However, under mild assumptions these are essentially in one-to-one correspondence with skewon-free tensors in $\Omega^2{}_2(N)$. The below presentation in Theorem 3.1 is based on the reformulations in Refs. 7 and 8.

Suppose L is an element in $\Omega^1(N) \otimes \Omega_1(N)$ on an n-manifold N. Then we can treat L as a pointwise linear map $\Omega^1(N) \to \Omega^1(N)$. By linear algebra, it follows that around each $p \in N$ there are coordinates such that at p, components $(L_i^j)_{ij}$ is a matrix in Jordan normal form. Since there are only finitely many ways an $n \times n$ matrix can be decomposed into Jordan blocks, it follows that there are only a finite number of normal forms for $L|_p$. It should be emphasised that the structure of the Jordan normal form is unstable under perturbations of the matrix. Hence, the normal form is in general only valid at one point. Also, for both a symbolic and numeric L, it can be difficult to determine the structure of the normal form.[22]

The normal form theorem in Ref. 38 is a normal form theorem for skewon-free elements κ in $\Omega^2{}_2(N)$ that is analogue to the Jordan normal form theorem for $\binom{1}{1}$-tensors. The difficulty in proving such a result is easy to understand. The matrix that represents κ at a point is a 6×6 matrix. By a linear transformation in $\mathbb{R}^6$, we can transform this into an Jordan normal form, but such a transformation, *a priori* has 36 degrees of freedom. On the other hand, for a coordinate transformation on N, the Jacobian only has 16 degrees of freedom. It is therefore not obvious that coordinate transformations have enough degrees of freedom to transform κ into a normal form. See Eq. (17). For a further discussion, see Ref. 8 and 38.

Theorem 3.1 below summarises the normal form theorem in Ref. 38 specialised to the setting that we need here. Let us make four comments. First, the below theorem is formulated for twisted $\kappa \in \widetilde{\Omega}^2{}_2(N)$ instead of for *area metrics* in Ref. 38 (which are ordinary tensors) or untwisted $\kappa \in \Omega^2{}_2(N)$ in Ref. 8. Second, the theorem is a consequence of the restatement in [Ref. 7, Theorem 1.6]. Third, the theorem contains the technical assumption that κ is invertible and the Fresnel surface has no two-dimensional subspace. This greatly simplifies the result since it implies that there are only 7 possible normal forms and one does not need any conjugations by Hodge operators. These assumptions will also appear in Theorem 5.1. For a further discussion of these assumptions, see end of Sec. V. Fourth, the reason the normal form theorem is useful can be seen from Proposition 2.1. Namely, in arbitrary coordinates, a skewon-free $\kappa \in \widetilde{\Omega}^2{}_2(N)$ depends on 21 parameters. However, from Theorem 3.1 we see that each normal form depends only on 2, 4 or 6 parameters. This reduction of parameters will make the computer algebra feasible in Theorem 5.1.

Theorem 3.1: *Suppose N is a 4-manifold and $\kappa \in \widetilde{\Omega}^2{}_2(N)$. Furthermore, suppose that at some $p \in N$*

(a) *κ has no skewon part at p,*
(b) *κ is invertible at p,*
(c) *the Fresnel surface $F_p(\kappa)$ does not contain a two dimensional vector subspace.*

Then there exists coordinates $\{x^i\}_{i=0}^3$ around p such that the 6×6 matrix$(\kappa_I^J)_{IJ}$ that represents $\kappa|_p$ in these coordinates is one of the below matrices:

- *Metaclass I:*

$$\begin{pmatrix} \alpha_1 & 0 & 0 & -\beta_1 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & -\beta_2 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & -\beta_3 \\ \beta_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \beta_3 & 0 & 0 & \alpha_3 \end{pmatrix} \tag{31}$$

- *Metaclass II:*

$$\begin{pmatrix} \alpha_1 & -\beta_1 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & -\beta_2 \\ 0 & 1 & 0 & \alpha_1 & \beta_1 & 0 \\ 1 & 0 & 0 & -\beta_1 & \alpha_1 & 0 \\ 0 & 0 & \beta_2 & 0 & 0 & \alpha_2 \end{pmatrix} \tag{32}$$

- *Metaclass III:*

$$\begin{pmatrix} \alpha_1 & -\beta_1 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 1 & 0 & \alpha_1 & 0 & 0 & -\beta_1 \\ 0 & 0 & 0 & \alpha_1 & \beta_1 & 1 \\ 0 & 0 & 1 & -\beta_1 & \alpha_1 & 0 \\ 0 & 1 & \beta_1 & 0 & 0 & \alpha_1 \end{pmatrix} \tag{33}$$

- *Metaclass IV:*

$$\begin{pmatrix} \alpha_1 & 0 & 0 & -\beta_1 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & -\beta_2 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_4 \\ \beta_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_4 & 0 & 0 & \alpha_3 \end{pmatrix} \tag{34}$$

- *Metaclass V:*

$$\begin{pmatrix} \alpha_1 & -\beta_1 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & \alpha_3 \\ 0 & 1 & 0 & \alpha_1 & \beta_1 & 0 \\ 1 & 0 & 0 & -\beta_1 & \alpha_1 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_2 \end{pmatrix} \tag{35}$$

- *Metaclass VI:*

$$\begin{pmatrix} \alpha_1 & 0 & 0 & -\beta_1 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & \alpha_4 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_5 \\ \beta_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \alpha_4 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_5 & 0 & 0 & \alpha_3 \end{pmatrix} \tag{36}$$

- *Metaclass VII:*

$$\begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & \alpha_5 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_6 \\ \alpha_4 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \alpha_5 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_6 & 0 & 0 & \alpha_3 \end{pmatrix} \tag{37}$$

In each matrix the parameters satisfy $\alpha_1, \alpha_2, \dots \in \mathbb{R}$, $\beta_1, \beta_2, \dots \in \mathbb{R} \setminus \{0\}$ *and* $\operatorname{sgn} \beta_1 = \operatorname{sgn} \beta_2 = \cdots$.

B. Medium with one Lorentz null cone

By a *pseudo-Riemann metric* on a manifold N we mean a symmetric $\binom{0}{2}$-tensor g that is non-degenerate. If N is not connected we also assume that g has constant signature. By a *Lorentz metric* we mean a pseudo-Riemann metric on a 4-manifold with signature $(-\ +\ +\ +)$ or $(+\ -\ -\ -)$. Let $\sharp$ be the isomorphisms $\sharp$: $T^*N \to TN$, so that if locally $g = g_{ij} dx^i \otimes dx^j$ then $\sharp(\alpha_i dx^i) = \alpha_i g^{ij} \frac{\partial}{\partial x^j}$. Using the $\sharp$-isomorphism we extend g to covectors by setting $g(\xi, \eta) = g(\xi^\sharp, \eta^\sharp)$ when $\xi, \eta \in T^*_p(N)$.

For a Lorentz metric g the *null cone* at a point $p \in N$ is defined as

$$N_p(g) = \{\xi \in T^*_p(N) : g(\xi, \xi) = 0\},$$

and analogously to the Fresnel surface we define $N(g) = \coprod_{p \in N} N_p(g)$.

If g is a pseudo-Riemann metric on a 4-manifold N, then the *Hodge star operator* of g is defined as the $*_g \in \tilde{\Omega}^2{}_2(N)$ such that if locally $g = g_{ij} dx^i \otimes dx^j$, and $*_g$ is written as in Eq. (4), then

$$(*_g)^{ij}_{rs} = \sqrt{|\det g|}\, g^{ia} g^{jb} \varepsilon_{abrs}, \tag{38}$$

where $\det g = \det g_{ij}$ and g^{ij} is the ijth entry of $(g_{ij})^{-1}$. Then $*_g$ has only a principal part. See, for example, Refs. 10 and 16. Moreover, if g is a Lorentz metric and $\kappa = *_g$, we have

$$F(\kappa) = N(g). \tag{39}$$

Equation (39) is the motivation for defining $N(g)$ as a subset of the cotangent bundle.

On $N = \mathbb{R}^4$, a specific example is $\kappa = \sqrt{\frac{\epsilon}{\mu}} *_g$, where g is the Lorentz metric $g = \operatorname{diag}(-\frac{1}{\epsilon\mu}, 1, 1, 1)$ on $\mathbb{R}^4$. Then constitutive equation (12) models standard isotropic medium on $\mathbb{R}^4$ with permittivity $\epsilon > 0$ and $\mu > 0$. When x^0 models time, this medium is an example of an *non-birefringent medium*. That is, the Fresnel surface $F_p(\kappa)$ has a single sheet, and there is only one signal speed in each spatial direction. In particular, propagation speed does not depend on polarisation.

Under some natural assumptions, the next theorem gives the complete characterisation and representation of medium tensors with one Lorentz null cone.

Theorem 3.2: *Suppose N is a 4-manifold. If $\kappa \in \widetilde{\Omega}^2{}_2(N)$ satisfies ${}^{(2)}\kappa = 0$, then the following conditions are equivalent*:

(i) ${}^{(3)}\kappa = 0$, $\det \kappa \neq 0$ *and there exists a Lorentz metric g such that Eq. (39) holds.*
(ii) $\kappa^2 = -f\,\mathrm{Id}$ *for some function $f \in C^\infty(N)$ with $f > 0$.*
(iii) there exists a Lorentz metric g and a non-vanishing function $f \in C^\infty(N)$ such that

$$\kappa = f *_g . \tag{40}$$

Implication (i) ⇒ (ii) was described in the introduction. Implication (iii) ⇒ (i) is a direct computation.

In the setting of electromagnetics, implication (ii) ⇒ (iii) seems to first to have been derived by Schönberg.[36,37] For further derivations and discussions, see Refs. 10, 16, 19, 28, 29, and 36. The above formulation is from Ref. 6.

When a general $\kappa \in \widetilde{\Omega}^2{}_2(N)$ on a 4-manifold N satisfies $\kappa^2 = -f\,\mathrm{Id}$ for a function $f \in C^\infty(N)$ one says that κ satisfies the *closure condition*. For physical motivation, see Ref. 16, Sec. D.3.1. For a study of more general closure relations, and in particular, for an analysis when κ might have a skewon part, see Refs. 10 and 24.

C. Medium with two Lorentz null cones

Since the Fresnel surface is a 4th order surface, the Fresnel surface can decompose into two distinct Lorentz null cones. In such a medium differently polarised waves can propagate with different wave speeds. This motivates the next definition.

Definition 3.3: Suppose N is a 4-manifold and $\kappa \in \widetilde{\Omega}^2{}_2(N)$. If $p \in N$ we say that the Fresnel surface $F_p(\kappa)$ *decomposes into two Lorentz null cones* if there exists two Lorentz metrics g_+ and g_- defined in a neighbourhood of p such that

$$F_p(\kappa) = N_p(g_+) \ \cup \ N_p(g_-) \tag{41}$$

and $N_p(g_+) \neq N_p(g_-)$.

If g, h are Lorentz metrics, then $N_p(g) \subset N_p(h)$ implies that at p we have $g = Ch$ for some $C \in \mathbb{R} \setminus \{0\}$. See, for example Ref. 40. Thus, if κ decomposes into two Lorentz null cones, then Eq. (39) cannot hold.

Under some assumptions, the next theorem (Theorem 2.1 in Ref. 7) gives the complete pointwise description of all medium tensors with two Lorentz null cones.

Theorem 3.4: *Suppose N is a 4-manifold and $\kappa \in \widetilde{\Omega}^2{}_2(N)$. Furthermore, suppose that at some $p \in N$*

(a) *κ has no skewon part at p,*
(b) *κ is invertible at p,*
(c) *the Fresnel surface $F_p(\kappa)$ decomposes into two Lorentz null cones.*

Then exactly one of the below three possibilities holds:

(i) ***Metaclass I.*** *There are coordinates $\{x^i\}_{i=0}^3$ around p such that the matrix $(\kappa_I^J)_{IJ}$ that represents $\kappa|_p$ in these coordinates is given by Eq. (31) for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ and $\beta_1, \beta_2, \beta_3 \in \mathbb{R} \setminus \{0\}$ with*

$$\alpha_2 = \alpha_3, \quad \beta_2 = \beta_3, \quad \operatorname{sgn}\beta_1 = \operatorname{sgn}\beta_2 = \operatorname{sgn}\beta_3$$

and either $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$ or both inequalities hold.

(ii) ***Metaclass II.*** *There are coordinates $\{x^i\}_{i=0}^3$ around p such that the matrix $(\kappa_I^J)_{IJ}$ that represents $\kappa|_p$ in these coordinates is given by Eq. (32) for some $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\beta_1, \beta_2 \in \mathbb{R} \setminus \{0\}$ with*

$$\alpha_1 = \alpha_2, \quad \beta_1 = \beta_2.$$

(iii) ***Metaclass IV.*** *There are coordinates $\{x^i\}_{i=0}^3$ around p such that the matrix $(\kappa_I^J)_{IJ}$ that represents $\kappa|_p$ in these coordinates is given by Eq. (34) for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ and $\beta_1, \beta_2 \in \mathbb{R}\setminus\{0\}$ with*

$$\alpha_1 = \alpha_2, \quad \beta_1 = \beta_2, \quad \alpha_4 \neq 0, \quad \alpha_3^2 \neq \alpha_4^2.$$

Conversely, if κ is defined by one of the above three possibilities, then the Fresnel surface of $\kappa|_p$ decomposes into two Lorentz null cones.

In Theorem 3.4, the class of uniaxial media is given by Metaclass I when $\alpha_1 = \alpha_2 = \alpha_3 = 0$. The main conclusion of the theorem is that there are two (and only two) additional medium classes where the Fresnel surface decomposes (Metaclasses II and IV). In all three classes, there are also explicit formulas for the Lorentz metrics that factorise the Fresnel surface. For a further discussion of these medium classes, see Ref. 7.

In Theorem 5.1 we will show that under suitable assumptions every skewon-free medium with two Lorentz null cones can be written as in Eq. (42). This medium class is a special class of *generalised Q-medium* introduced by Lindell and Wallén in Ref. 25. For further discussions of this medium class, see Refs. 10, 24, and 26.

Proposition 3.5: Suppose N is a 4-manifold, g is a Lorentz metric, ρ is a twisted scalar density of weight 1, $A \in \Omega_2(N)$, and $C_1 \in \mathbb{R}\setminus\{0\}$ and $C_2 \in \mathbb{R}$. Moreover, suppose $\kappa \in \widetilde{\Omega}^2{}_2(N)$ is defined as

$$\kappa = C_1 *_g \; + \rho \widehat{A} \otimes A \; + \; C_2 \,\mathrm{Id}\,. \tag{42}$$

Then κ is skewon-free. If $\kappa|_p$ is invertible for some $p \in N$ then:

(i) *There exists a Lorentz metric g' such that $F_p(\kappa) = N_p(g')$ if and only if $A|_p = 0$ or $\rho|_p = 0$.*

(ii) *$\kappa|_p$ decomposes into two Lorentz null cones if and only if $\rho|_p \neq 0$, $A|_p \neq 0$ and at p we have*

$$\det \kappa \neq \left(C_1^2 + C_2^2\right)^2 \left(C_2 + \frac{1}{2}\,\mathrm{trace}(\rho\, \widehat{A} \otimes A)\right)^2. \tag{43}$$

Moreover, when equivalence holds in (i), then $F_p(\kappa) = N_p(g)$, and when equivalence holds in (i), then $F_p(\kappa) \supset N_p(g)$.

Proof: Let $\{x^i\}_{i=0}^3$ be coordinates around p such that the Lorentz metric has components $g = \pm\,\mathrm{diag}(-1, 1, 1, 1)$ at p. In what follows all computations will be done at p. For claim *(i)*, let us note that the axion component of κ does not influence the Fresnel polynomial. See, for example, Ref. 16. Thus Eq. (39) holds when $A = 0$ or $\rho = 0$. For the converse direction, suppose $F_p(\kappa) = N_p(g')$ for some Lorentz metric g'. Then Theorem 3.2 implies that $(\kappa - \frac{1}{6}\,\mathrm{trace}\,\kappa\;\mathrm{Id})^2 = -\lambda\,\mathrm{Id}$ for some $\lambda > 0$. Writing out the last equation and solving the associated Gröbner basis equations (see Refs. 5 and 6) shows that $A = 0$ or $\rho = 0$. For claim *(ii)*, let us write $A = \frac{1}{2} A^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$. Then the Fresnel polynomial at p is given by

$$\mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l = -C_1^2 \left(g^{ij}\xi_i\xi_j\right)\left(H^{ij}\xi_i\xi_j\right), \tag{44}$$

where $g^{ij} = (g^{-1})_{ij}$ and $H^{ij} = C_1 g^{ij} - 2\rho A^{ia} g_{ab} A^{bj}$ (see Refs. 24 and 25). Moreover,

$$\det \kappa = \left(C_1^2 + C_2^2\right)^2 \left(C_1^2 + C_2^2 + E + C_2\,\mathrm{trace}(\rho\,\widehat{A} \otimes A)\right), \tag{45}$$

where $E \in \mathbb{R}$ is an expression that depends on ρ, C_1, and A. We will not need the explicit expression for E. However, by computer algebra we see that the same E also appears in $\det H$ for matrix $H = (H^{ij})_{ij}$. Then Eq. (45) yields

$$\begin{aligned}\det H &= -\left(C_1^2 + E - \frac{1}{4}\left(\mathrm{trace}(\rho\,\widehat{A} \otimes A)\right)^2\right)^2 \\ &= -\left(\frac{\det \kappa}{(C_1^2 + C_2^2)^2} - \left(C_2 + \frac{1}{2}\,\mathrm{trace}(\rho\,\widehat{A} \otimes A)\right)^2\right)^2.\end{aligned} \tag{46}$$

If κ decomposes into two Lorentz null cones, claim (i) implies that $A \neq 0$ and $\rho \neq 0$. Moreover, by Proposition 1.5 in Ref. 7 and since polynomials have a unique factorisation into irreducible factors (Ref. 5, Theorem 5 in Sec. 3.5), we have $\det H < 0$ and Eq. (46) implies inequality (43) for $\det \kappa$. Conversely, if the inequalities in claim (ii) are satisfied, then Eq. (46) shows that $\det H < 0$, so g and H both have Lorentz signature at p. To complete the proof we need to show that there is no constant $C \in \mathbb{R} \setminus \{0\}$ such that $g^{ij} = CH^{ij}$. Since $A \neq 0$ and $\rho \neq 0$, this follows by inspecting equations $g^{ii} = CH^{ii}$ for $i = 0, \dots, 3$. □

IV. DECOMPOSABLE MEDIA

In this section we first describe the class of decomposable media introduced in Ref. 24. In particular, in Theorem 4.3 we describe the sufficient conditions derived in Ref. 24 that imply that a medium is decomposable. In Theorem 5.1 these conditions will play a key role. In Sec. IV C we will describe some results that suggest that condition *(i)* in Theorem 4.3 is, in fact, a general factorisability condition for the Fresnel polynomial. Following Ref. 24 we restrict the analysis to $\mathbb{R}^4$. This will simplify the analysis since we can work with plane waves. However, let us emphasise that condition *(i)* in Theorem 4.3 naturally generalises into a tensorial condition.

A. Plane waves in $\mathbb{R}^4$

We say that a tensor T on $\mathbb{R}^4$ is *constant* if there are global coordinates for $\mathbb{R}^4$ where the components for T are constant. If we assume that many tensors are constant, we assume that they are constant with respect to the same choice of coordinates. Below we also use notation $\Omega^k(N, \mathbb{C})$ to denote the space of k-forms on a manifold N with possibly complex coefficients.

Suppose $\kappa \in \Omega^2{}_2(\mathbb{R}^4)$ is constant and $F, G \in \Omega^2(\mathbb{R}^4)$ are defined as

$$F = \mathrm{Re}\{e^{i\Phi} X\}, \quad G = \mathrm{Re}\{e^{i\Phi} Y\}, \tag{47}$$

where Φ is a function $\Phi : \mathbb{R}^4 \to \mathbb{R}$ such that $d\Phi$ is constant and non-zero, $X, Y \in \Omega^2(\mathbb{R}^4, \mathbb{C})$ are constant and not both zero. If F and G solve the sourceless Maxwell's equations we say that F and G is a *plane wave*.

Proposition 4.1: Suppose $\kappa \in \Omega^2{}_2(\mathbb{R}^4)$ is constant and Φ is a function $\Phi : \mathbb{R}^4 \to \mathbb{R}$ such that $d\Phi$ is constant and non-zero. Moreover, suppose X, Y are constant 2-forms $X, Y \in \Omega^2(\mathbb{R}^4, \mathbb{C})$. If F and G are defined by Eq. (47)*, then the following conditions are equivalent:*

(i) *F and G is a plane wave.*
(ii) *$d\Phi \in F(\kappa)$ and there exists a constant $\alpha \in \Omega^1(\mathbb{R}^4, \mathbb{C})$ such that $d\Phi \wedge \alpha \neq 0$, $d\Phi \wedge \kappa(d\Phi \wedge \alpha) = 0$ and*

$$X = d\Phi \wedge \alpha, \tag{48}$$

$$Y = \kappa(d\Phi \wedge \alpha). \tag{49}$$

Proof: Let $\xi = d\Phi$. If F and G is a plane wave then $\xi \neq 0$ implies that

$$\xi \wedge X = 0, \quad \xi \wedge Y = 0, \quad Y = \kappa(X). \tag{50}$$

The first equation in Eq. (50) implies that there exists a constant 1-form $\alpha \in \Omega^1(\mathbb{R}^4, \mathbb{C})$ such that $X = \xi \wedge \alpha$. It is clear that α and $\xi \wedge \alpha$ are both non-zero, since otherwise $X = Y = 0$. Combining the latter two equations in Eq. (50) implies that

$$\xi \wedge \kappa(\xi \wedge \alpha) = 0. \tag{51}$$

Since this linear equation for α has a non-zero solution, it follows that $\xi \in F(\kappa)$. See, for example, Refs. 6,16,28, and 36. This completes the proof of implication (i) $\Rightarrow$ (ii). For the converse implication it suffices to verify that Eqs. (47)–(49) define a solution to Maxwell's equations. □

B. Decomposable medium

The next definition and theorem are from Ref. 24. It is not known if the converse of Theorem 4.3 is also true.[24]

Definition 4.2: Suppose $\kappa \in {\Omega^2}_2(\mathbb{R}^4)$ is constant. Then we say that κ is *decomposable* if there exist non-zero and constant $A, B \in \Omega_2(\mathbb{R}^4)$ such that if F, G is a plane wave solution to Maxwell's equations, then

$$F(A) = 0 \quad \text{or} \quad F(B) = 0. \tag{52}$$

Theorem 4.3: *Suppose $\kappa \in {\Omega^2}_2(\mathbb{R}^4)$ is constant. Furthermore, suppose*

(i) *there exist constant tensors $A, B \in \Omega_2(\mathbb{R}^4)$ and a constant scalar density ρ of weight 1 such that*

$$\alpha \,\mathrm{Id} + \beta\,(\kappa + \overline{\kappa}) + \gamma \overline{\kappa} \circ \kappa = \rho \left(\widehat{A} \otimes B + \widehat{B} \otimes A\right) \tag{53}$$

for constants $\alpha, \beta, \gamma \in \mathbb{R}$ and β, γ are not both zero.

(ii) *the right hand side in Eq. (53) is non-zero.*

Then κ is decomposable (and condition (52) holds for the same A and B as in condition (53)).

Let us note that by Lemma 2.3, the right-hand side in Eq. (53) is non-zero if and only if A, B and ρ are all non-zero. For a proof of Theorem 4.3 see Ref. 24.

In Theorem 5.1 we will see that all the medium tensors in Theorem 3.4 are decomposable. In particular, every uniaxial medium tensor is decomposable. The next proposition shows that a slight generalisation of the class of isotropic media contains no decomposable medium tensors.

Proposition 4.4: Suppose $\kappa \in {\Omega^2}_2(\mathbb{R}^4)$ is defined as

$$\kappa = C_1 *_g \; + C_2 \,\mathrm{Id},$$

where $C_1 \in \mathbb{R} \setminus \{0\}$, $C_2 \in \mathbb{R}$ and g is a constant indefinite pseudo-Riemann metric on $\mathbb{R}^4$. Then κ is not decomposable.

Proof: Let us first assume that g is a Lorentz metric and let $\{x^i\}_{i=0}^3$ be coordinates such that $g = k\,\mathrm{diag}(-1, 1, 1, 1)$ for some $k \in \{-1, 1\}$. At $0 \in \mathbb{R}^4$, it follows that

$$F_0(\kappa) = \{\xi \in T_0^*(\mathbb{R}^4) : -\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 0\}.$$

For a contradiction, suppose κ is decomposable. By Proposition 4.1 there exist non-zero and constant $A, B \in \Omega_2(\mathbb{R}^4)$ such that

$$(\xi \wedge \alpha)(A)\ (\xi \wedge \alpha)(B) = 0 \tag{54}$$

for all $\xi, \alpha \in T_0^*(\mathbb{R}^4)$ that satisfy $\xi \in F_0(\kappa)$ and

$$\xi \wedge \alpha \neq 0, \quad \xi \wedge \kappa(\xi \wedge \alpha) = 0. \tag{55}$$

Let G is the subset $G \subset F_0(\kappa) \setminus \{0\}$ for which each coordinate belongs to $\{0, 1, \sqrt{2}, \sqrt{3}\}$. That is, one can think of G as a discretisation of $F_0(\kappa)$ in one quadrant of $T_0^*(\mathbb{R}^4)$. In total there are 19 such points, and for each $\xi \in G$, we can find two linearly independent $\alpha \in T_0^*(\mathbb{R}^4)$ such that conditions (55) holds, cf. Ref. 6. Insisting that Eq. (54) holds for all such ξ and α gives $19 \times 2 = 38$ second order polynomial equations for the coefficients in A and B. Computing a Gröbner basis for these equations and solving implies that either $A = 0$ or $B = 0$. See Ref. 5. Hence κ is not decomposable. When g has signature $(-\ -\ +\ +)$ the claim follows by repeating the above argument. □

C. Factorisability of the Fresnel polynomial

In what follows condition (i) in Theorem 4.3 will play a key role. Let us therefore introduce the following definition.

Definition 4.5: If $\kappa \in \Omega^2{}_2(\mathbb{R}^4)$ is constant and satisfies condition (i) in Theorem 4.3, then we say that κ is *algebraically decomposable*.

In Ref. 24, Lindell, Bergamin, and Favaro showed that if κ is algebraically decomposable (plus some additional assumptions), then the Fresnel polynomial of κ always factorises into the product of two quadratic forms. In this section we summarise this result in Theorem 4.6. Moreover, we will see that for algebraically decomposable medium, the Fresnel polynomial seems to factorise even when the additional assumptions in Theorem 4.6 are not satisfied. These results suggest (but do not prove) that the definition of algebraically decomposable medium might be a sufficient condition for the Fresnel polynomial to factorise.

Let us first note that the class of algebraically decomposable media contains a number medium classes as special cases. Suppose $\kappa \in \Omega^2{}_2(\mathbb{R}^4)$. If κ is purely skewon, then $\kappa + \overline{\kappa} = 0$ and κ is algebraically decomposable. Also, if κ satisfies the *mixed closure condition* $\overline{\kappa} \circ \kappa = \lambda\,\mathrm{Id}$,[10,24] then κ is algebraically decomposable. If κ has no skewon part, then $\overline{\kappa} = \kappa$ and the definition of algebraically decomposable medium simplifies. Thus, if κ has no skewon part and if κ is a *self-dual medium* (so that $\alpha\,\mathrm{Id} + \beta\kappa + \gamma\kappa^2 = 0$),[23] then κ is algebraically decomposable. In particular, a skewon-free medium κ that satisfies the *closure condition* $\kappa^2 = \lambda\,\mathrm{Id}$ (see Ref. 16) is algebraically decomposable.

Equation (53) that defines algebraically decomposable medium is a nonlinear equation in κ. Suppose $\{x^i\}_{i=0}^3$ are coordinates for $\mathbb{R}^4$, $P \in \mathbb{R}^{6\times 6}$ is the matrix $P = (\kappa_I^J)_{IJ}$ that represents κ and $A, B \in \mathbb{R}^6$ are the column vectors $A = (A^I)_I$ and $B = (B^I)_I$ that represent bivectors A and B with components as in Sec. II D. Then Eq. (53) reads

$$\alpha E + \beta(P^t E + E P) + \gamma P^t E P = 2\rho(A B^t + B A^t), \tag{56}$$

where A^t is the matrix transpose and $E \in \mathbb{R}^{6\times 6}$ is the matrix $E = (\varepsilon^{IJ})_{IJ}$. Numerically, $E = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, where 0 and I are the zero and identity 3×3 matrices. When $\gamma \neq 0$, Eq. (56) is structurally similar to an *algebraic Riccati equation*.[12]

The next theorem summarises the factorisation result from Ref. 24, but restated in the present setting.

Theorem 4.6: *If $\kappa \in \Omega^2{}_2(\mathbb{R}^4)$ is algebraically decomposable and $\alpha, \beta, \gamma, \rho, A, B$ in Eq. (53) satisfy one of the below conditions*:

(i) $\gamma = 0$,
(ii) $\gamma \neq 0$, $\beta^2 - \alpha\gamma \neq 0$ *and there exists a* $D \in \Omega_2(\mathbb{R}^4)$ *such that*

$$D\,(\gamma\kappa + \beta\,\mathrm{Id}) = \frac{1}{2}\,\mathrm{trace}(\rho\,\widehat{D} \otimes D)\,A + \gamma B. \tag{57}$$

Then the Fresnel polynomial of κ factorises into the product of two quadratic forms.

Let us note that Eq. (57) is a nonlinear equation for D. *A priori*, the equation has real solutions, complex solutions, or no solutions for D. For a discussion of the last possibility, see below. Pointwise $\mathrm{trace}(\rho\,\widehat{D} \otimes D) = 0$ holds if and only if $D \wedge D = 0$ or $\rho = 0$.

Let us outline the argument in Ref. 24 used to prove Theorem 4.6. Suppose $\Omega^2{}_2(\mathbb{R}^4)$ is algebraically decomposable. If assumption (i) holds, then by rescaling we may assume that $\beta = 1$. Then, since $\kappa + \overline{\kappa} = 2({}^{(1)}\kappa + {}^{(3)}\kappa)$, it follows that

$$\alpha\,\mathrm{Id} + 2(\kappa - \sigma) = \rho\left(\widehat{A} \otimes B + \widehat{B} \otimes A\right) \tag{58}$$

for some $\sigma \in \Omega^2{}_2(\mathbb{R}^4)$ with only a skewon part. This gives an explicit representation formula for all κ that satisfy condition (53) with $\gamma = 0$. Computing the Fresnel polynomial for κ shows that it factorises into two quadratic forms. On the other hand, when assumption (ii) holds, then Theorem 4.7 in the below shows that Eq. (53) transforms into $\overline{\eta} \circ \eta = \lambda\,\mathrm{Id}$ for some $\lambda \neq 0$ by a transformation similar to completing the square. Thus, to understand the structure of algebraically decomposable medium that satisfy assumption (ii), we only need to understand the simpler equation $\overline{\eta} \circ \eta = \lambda\,\mathrm{Id}$ with $\lambda \neq 0$. In[24] the latter equation is solved (see also Ref. 10) using two explicit representation formulas similar to Eq. (58). Using these representation formulas, the Fresnel polynomial can again be computed, and in both cases it factorises into a product of quadratic forms.

The next theorem from Ref. 24 describes the transformation property of Eq. (53) used in the proof of Theorem 4.6. The proof is a direct computation using identities (22)–(25). For a general discussion of transformation properties for the matrix algebraic Riccati equation, see Refs. 3 and 21.

Theorem 4.7: *Suppose $\kappa \in \Omega^2{}_2(\mathbb{R}^4)$ is algebraically decomposable such that Eq. (53) holds with $\gamma \neq 0$. If, moreover, there exists a $D \in \Omega_2(\mathbb{R}^4)$ such that Eq. (57) holds, then $\eta \in \Omega^2{}_2(\mathbb{R}^4)$ defined as*

$$\eta = \gamma\kappa - \rho\widehat{D} \otimes A + \beta\,\mathrm{Id} \tag{59}$$

satisfies

$$\overline{\eta} \circ \eta = (\beta^2 - \alpha\gamma)\,\mathrm{Id}\,. \tag{60}$$

Suppose κ is algebraically decomposable such that Eq. (53) holds with $\gamma \neq 0$ and $\beta^2 - \alpha\gamma = 0$. Now we cannot use Theorem 4.6 to decise whether the Fresnel polynomial factorises. However, by computer algebra we can find explicit examples of medium tensors with the above properties. Preliminary computer algebra experiments using such expressions suggest that the Fresnel polynomial always seems to factorise also in this case. However, the factorisation seems be qualitatively different. Condition $\beta^2 - \alpha\gamma = 0$ seems to imply a linear factor in the Fresnel polynomial. For example, the Fresnel polynomial can factorise into the product of irreducible 1st and 3rd order polynomials. On the other hand, suppose κ is algebraically decomposable such that Eq. (53) holds with $\gamma \neq 0$, $\beta^2 - \alpha\gamma \neq 0$ and Eq. (57) has no real solution for D. Now we can neither use Theorem 4.6 do decise whether the Fresnel polynomial factorises, but we may again construct explicit examples of medium tensors with the above properties. Using these expressions, preliminary computer algebra experiments suggest that the Fresnel polynomial also seems to factorise in this case. In conclusion, these initial observations together with Theorem 4.6 suggest that the definition of algebraically decomposable medium could be a sufficient condition for the Fresnel polynomial to factorise.

Lastly, let us note that algebraic Riccati equations, and more generally, quadratic matrix equations, appear in a number of fields. In view of Theorem 4.6 and Eq. (56), it is, however, interesting to note that quadratic matrix equations appear in the study of polynomial factorisation in one variable.[1] Differential Riccati equations also appear in the problem of factoring linear partial differential operators of second and third order.[14]

V. CHARACTERISATION AND REPRESENTATION OF MEDIUM TENSORS WITH TWO LORENTZ NULL CONES

As described in the Introduction, the next theorem is the main result of this paper. A discussion of the theorem is postponed to the end of this section.

Theorem 5.1: *Suppose N is a 4-manifold and $\kappa \in \widetilde{\Omega}^2{}_2(N)$ is skewon-free and invertible at a point $p \in N$. Then the following conditions are equivalent:*

(i) The Fresnel surface of κ decomposes into two Lorentz null cones at p.

(ii) κ satisfies conditions:

(a) the Fresnel surface $F_p(\kappa) \subset T^*_p(N)$ does not contain a two-dimensional vector subspace.

(b) there are $A, B \in \Omega^2(N)$ and a tensor density ρ of weight 1 such that at p we have

$$(\kappa + \mu\,\mathrm{Id})^2 = -\lambda\,\mathrm{Id} + \rho\left(\widehat{A} \otimes B + \widehat{B} \otimes A\right) \tag{61}$$

for some $\mu \in \widetilde{C}^\infty(N)$ and $\lambda \in C^\infty(N)$. Moreover, $A, B, \rho \neq 0$, and $\lambda > 0$ at p.

(iii) *Around p there is a locally defined Lorentz metric g, a locally defined non-zero twisted scalar density ρ of weight 1, an $A \in \Omega_2(N)$ that is non-zero at p, and constants $C_1 \in \mathbb{R} \setminus \{0\}$ and $C_2 \in \mathbb{R}$ such that at p,*

$$\kappa = C_1 *_g + \rho \widehat{A} \otimes A + C_2 \operatorname{Id}, \tag{62}$$

and κ satisfies inequality (43) at p.

Moreover, when equivalence holds, then $N_p(g) \subset F_p(\kappa)$ when g is the Lorentz metric in Eq. (62).

In the Theorem 5.1 we will use the computer algebra technique of *Gröbner bases*[5] to eliminate variables from polynomial equations. This technique was also used in Ref. 7. Let $\mathbb{C}[u_1, \dots, u_N]$ the ring of complex coefficient polynomials $\mathbb{C}^N \to \mathbb{C}$ in variables $u_1, \dots, u_N$. For polynomials $r_1, \dots, r_k \in \mathbb{C}[u_1, \dots, u_N]$, let

$$\langle r_1, \dots, r_k \rangle = \{\sum_{i=1}^{k} f_i r_i : f_i \in \mathbb{C}[u_1, \dots, u_N]\}$$

be the *ideal generated by $r_1, \dots, r_k$*. Suppose $V \subset \mathbb{C}^N$ is the solution set to polynomial equations $p_1 = 0, \dots, p_M = 0$ where $p_i \in \mathbb{C}[u_1, \dots, u_N]$. If I is the ideal generated by $p_1, \dots, p_M$, the *elimination ideals* are the ideals defined as

$$I_k = I \cap \mathbb{C}[u_{k+1}, \dots, u_N], \quad k \in \{0, \dots, N-1\}.$$

Thus, if $(u_1, \dots, u_N) \in V$ then by Ref. 5, Proposition 9, Sec. 2.5 it follows that $p(u_{k+1}, \dots, u_N) = 0$ for any $p \in I_k$, and I_k contain polynomial consequences of the original equations that only depend on variables $u_{k+1}, \dots, u_N$. Using Gröbner basis, one can explicitly compute I_k as mentioned in Ref. 5, Theorem 2 in Sec. 3.1. In the below proof this has been done with the built-in Mathematica routine "GroebnerBasis." The same technique of eliminating variables was also a key part of the proof of Theorem 3.4 in Ref. 7.

Proof: Let us first prove implication (i) ⇒ (ii). By Ref. 7, Proposition 1.3 condition (i) implies that $F_p(\kappa)$ has no two dimensional subspace. By Theorem 3.4 we only need to check three medium classes.

Metaclass I. If $\kappa|_p$ is in Metaclass I, then κ can be written as in Eq. (31) with conditions on the parameters given by Theorem 3.4. Suppose $\alpha_1 = \alpha_2$. Then Theorem 3.4 implies that $\beta_1 \neq \beta_2$. Let $\rho = \frac{1}{2}(\beta_2^2 - \beta_1^2)$, $\mu = -\alpha_1$, $\lambda = \beta_2^2$. Moreover, let A and B be bivectors defined as $A = \frac{1}{2} A^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$ and similarly for B, with coefficients

$$(A^{ij})_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}, \quad (B^{ij})_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}, \tag{63}$$

where subdiagonal terms are determined by antisymmetry. For these parameters, computer algebra shows that Eq. (61) holds. On the other hand, if $\alpha_1 \neq \alpha_2$, suitable parameters are

$$\rho = \frac{1}{8(\alpha_1 - \alpha_2)\beta_1}, \quad \mu = -\alpha_2, \quad \lambda = \beta_2^2,$$

and

$$(A^{ij})_{ij} = \begin{pmatrix} 0 & 2(\alpha_1 - \alpha_2)\beta_1 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & (\alpha_1 - \alpha_2)^2 - \beta_1^2 + \beta_2^2 + \sqrt{\sigma} \\ & & & 0 \end{pmatrix},$$

where

$$\sigma = \left((\alpha_1-\alpha_2)^2+(\beta_1-\beta_2)^2\right)\left((\alpha_1-\alpha_2)^2+(\beta_1+\beta_2)^2\right).$$

Bivector B is defined by the same formula as for A, but by replacing $\sqrt{\sigma}$ with $-\sqrt{\sigma}$.

Metaclass II. If $\kappa|_p$ is in Metaclass II, then κ can be written as in Eq. (32) with conditions on the parameters given by Theorem 3.4. Suitable parameters are $\rho = \beta_1/2$, $\mu = -\alpha_1$, $\lambda = \beta_1^2$, and

$$(A^{ij})_{ij} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}, \quad (B^{ij})_{ij} = \begin{pmatrix} 0 & 1 & -1 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}. \tag{64}$$

Metaclass IV. If $\kappa|_p$ is of Metaclass IV, then κ can be written as in Eq. (34) with conditions on the parameters given by Theorem 3.4. If $\alpha_1 \neq \alpha_3$, then suitable parameters are

$$\rho = \frac{1}{8(\alpha_3-\alpha_1)\alpha_4}, \quad \mu = -\alpha_1, \quad \lambda = \beta_1^2$$

and

$$(A^{ij})_{ij} = \begin{pmatrix} 0 & 0 & 0 & (\alpha_1-\alpha_3)^2+\alpha_4^2+\beta_1^2+\sqrt{\sigma} \\ & 0 & 2(\alpha_3-\alpha_1)\alpha_4 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix},$$

where

$$\sigma = \left(\alpha_4^2-(\alpha_3-\alpha_1)^2\right)^2+\beta_1^2\left(2\alpha_4^2+\beta_1^2+2(\alpha_1-\alpha_2)^2\right).$$

and B is defined as in Metaclass I. On the other hand, if $\alpha_1 = \alpha_3$, then suitable parameters are $\rho = \frac{1}{2}(\beta_1^2+\alpha_4^2)$, $\mu = -\alpha_3$, $\lambda = \beta_1^2$ and

$$(A^{ij})_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 1 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}, \quad (B^{ij})_{ij} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}. \tag{65}$$

This completes the proof of implication (i)$\Rightarrow$ (ii).

For the converse implication (ii) $\Rightarrow$ (i), suppose that κ satisfies the conditions in (ii). By Theorem 3.1 we may assume that there are coordinates $\{x^i\}_{i=0}^3$ around p such that at p, tensor κ is given by one of the matrices in Eqs. (31)–(37) for some parameters as in Theorem 3.1. Let us consider each of the seven cases separately.

Metaclass I. If $\kappa|_p$ is in Metaclass I, then there are coordinates $\{x^i\}_{i=0}^3$ around p such that κ is given by Eq. (31). By scaling A and B we may assume that $\rho|_p = 1$. Moreover, writing out Eq. (61) and eliminating variables in A and B using a Gröbner basis (see above) yields equations that only involve λ, μ and the parameters in κ. The rest of the argument is divided into three subcases:

Case 1: If $\beta_1 = \beta_2 = \beta_3$ the Gröbner basis equations imply that $\lambda = \beta_1^2$ and

$$(\alpha_2+\mu)(\alpha_3+\mu) = 0, \tag{66}$$

$$(\alpha_1+\mu)(\alpha_3+\mu) = 0, \tag{67}$$

$$(\alpha_1+\mu)(\alpha_2+\mu) = 0. \tag{68}$$

It follows that $\alpha_1, \alpha_2, \alpha_3$ cannot be all distinct, and by a coordinate change, we may assume that $\alpha_2 = \alpha_3$. If $\alpha_1 = \alpha_2 = \alpha_3$, Eq. (66) implies that $\mu = -\alpha_1$. Then Eq. (31) implies that

$\kappa = -\beta_1 *_g + \alpha_1 \,\mathrm{Id}$ at p, where g is the Hodge star operator for the locally defined Lorentz metric $g = \mathrm{diag}(-1, 1, 1, 1)$. Then Eq. (61) implies that $\rho\,(\widehat{A} \otimes B + \widehat{B} \otimes A) = 0$. Since this contradicts Lemma 2.3, we have $\alpha_1 \neq \alpha_2$ and κ decomposes into two Lorentz null cones at p by Theorem 3.4.

Case 2: If exactly two of β_1, β_2, β_3 coincide, then after a coordinate change we may assume that $\beta_1 \neq \beta_2 = \beta_3$. Then the Gröbner basis equations imply that either $\lambda = \beta_1^2$ or $\lambda = \beta_2^2$. If $\lambda = \beta_1^2$, the Gröbner basis equations imply that $\alpha_1 = \alpha_2 = \alpha_3$ and $\beta_1 = \beta_2 = \beta_3$. We may therefore assume that $\lambda = \beta_2^2$. Then the Gröbner basis equations imply that $\mu = -\alpha_2 = -\alpha_3$, and κ decomposes into two Lorentz null cones at p by Theorem 3.4.

Case 3: If all β_1, β_2, β_3 are all distinct, then the Gröbner basis equations imply that

$$(\beta_2^2 - \lambda)(\beta_3^2 - \lambda)(\alpha_1 + \mu) = 0,$$
$$(\beta_1^2 - \lambda)(\beta_3^2 - \lambda)(\alpha_2 + \mu) = 0,$$
$$(\beta_1^2 - \lambda)(\beta_2^2 - \lambda)(\alpha_3 + \mu) = 0,$$
$$(\beta_1^2 - \lambda)(\beta_2^2 - \lambda)(\beta_3^2 - \lambda) = 0.$$

These equations imply that we must have $\lambda = \beta_i^2$ and $\mu = -\alpha_i$ for some $i \in \{1, 2, 3\}$. If $i = 1$ the Gröbner basis equations imply that $\alpha_1 = \alpha_2 = \alpha_3$ and $\beta_1 = \beta_2$. This contradicts the assumption that all β_i are distinct. Similarly, $i = 2$ and $i = 3$ lead to contradictions, and Case 3 is not possible.

Metaclass II. If $\kappa|_p$ is in Metaclass II, there are coordinates $\{x^i\}_{i=0}^3$ around p such that κ is given by Eq. (32). Writing out Eq. (61) and eliminating variables as in Metaclass I gives equations that only involve variables λ, μ and the variables in κ. Solving these equations give

$$\mu = -\alpha_2, \quad \lambda = \beta_2^2, \quad \beta_1 = \beta_2, \quad \alpha_1 = \alpha_2,$$

and κ decomposes into two Lorentz null cones at p by Theorem 3.4.

Metaclass III. If $\kappa|_p$ is in Metaclass III, there are coordinates $\{x^i\}_{i=0}^3$ around p such that κ is given by Eq. (33). Eliminating variables as in Metaclass I implies that $\beta_1 = 0$. Thus $\kappa|_p$ cannot be in Metaclass III.

Metaclass IV. If $\kappa|_p$ is in Metaclass IV, there are coordinates $\{x^i\}_{i=0}^3$ around p such that κ is given by Eq. (34). We have $\alpha_4 \neq 0$ since otherwise $\mathrm{span}\{dx^1|_p, dx^2|_p\} \subset F_p(\kappa)$. Moreover, since κ is invertible at p it follows that $\alpha_3^2 \neq \alpha_4^2$. Writing out Eq. (61), eliminating variables as in Metaclass I, and solving implies that

$$\lambda = \beta_1^2, \quad \beta_1 = \beta_2, \quad \mu = -\alpha_1, \quad \alpha_1 = \alpha_2,$$

and κ decomposes into two Lorentz null cones at p by Theorem 3.4.

Metaclass V. If $\kappa|_p$ is in Metaclass V, there are coordinates $\{x^i\}_{i=0}^3$ around p such that κ is given by Eq. (35). We may assume that $\alpha_3 \neq 0$, since otherwise $\mathrm{span}\{dx^i|_p\}_{i=1}^3 \subset F_p(\kappa)$. Eliminating variables as in Metaclass I, and solving implies the contradiction $\lambda + \alpha_3^2 = 0$. Since $\lambda > 0$ it follows that $\kappa|_p$ cannot be in Metaclass V.

Metaclass VI. If $\kappa|_p$ is in Metaclass VI, there are coordinates $\{x^i\}_{i=0}^3$ around p such that κ is given by Eq. (36). Eliminating variables as in Metaclass I implies that

$$\left(\lambda + \alpha_5^2 + (\alpha_3 + \mu)^2\right)\left(\lambda + (\alpha_2 - \alpha_4 + \mu)^2\right)\left(\lambda + (\alpha_2 + \alpha_4 + \mu)^2\right) = 0.$$

Since $\lambda > 0$, it follows that $\kappa|_p$ cannot be in Metaclass VI.

Metaclass VII. If $\kappa|_p$ is in Metaclass VII, there are coordinates $\{x^i\}_{i=0}^3$ around p such that κ is given by Eq. (37). Eliminating variables as in Metaclass I and solving implies that

$$\prod_{k=1}^{3} \left(\lambda + \alpha_{k+3}^2 + (\alpha_k + \mu)^2\right) = 0.$$

Since $\lambda > 0$, it follows that $\kappa|_p$ cannot be in Metaclass VII. This completes the proof of implication (ii) $\Rightarrow$ (i).

Implication (iii) $\Rightarrow$ (i) is a restatement of Proposition 3.5. To prove implication (i) $\Rightarrow$ (iii) we proceed as in implication *(i)* $\Rightarrow$ *(ii)* and by Theorem 3.4 we only need to check three medium classes. Also, by Proposition 3.5 we do not need to prove inequality (43) since it follows from the other conditions in (iii) when (i) holds.

Metaclass I. If $\kappa|_p$ is in Metaclass I, there are coordinates $\{x^i\}_{i=0}^3$ around p such that κ is given by Eq. (31) with conditions on the parameters given by Theorem 3.4. Suppose $\alpha_1 = \alpha_2$. Let $C_1 = -\frac{\beta_2^2}{\Psi\sqrt{|\det g|}}$, $C_2 = \alpha_2$, $\Psi = \frac{\beta_2^2}{\beta_1}$ and in coordinates $\{x^i\}$, let ρ be defined by $\rho = (\beta_2^2 - \beta_2^1)/(2\beta_1)$. Then Eq. (62) holds when $A = \frac{1}{2}B^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$ when coefficients B^{ij} are as in Eq. (63) and g is the Lorentz metric $g = g_{ij}dx^i \otimes dx^j$ with coefficients

$$(g_{ij})_{ij} = \left(\operatorname{diag}\left(1, -1, -\frac{\Psi}{\beta_2}, -\frac{\Psi}{\beta_2}\right)\right)^{-1}. \tag{69}$$

On the other hand, suppose $\alpha_1 \neq \alpha_2$. Let Ψ be one of the two roots to the quadratic equation

$$\frac{1}{\beta_2}\Psi^2 - D_3\Psi + \beta_2 = 0, \tag{70}$$

where D_3 is defined as in Ref. 7, Theorem 2.1 (i)

$$D_3 = \frac{(\alpha_1 - \alpha_2)^2 + \beta_1^2 + \beta_2^2}{\beta_1\beta_2}.$$

Since $\operatorname{sgn}\beta_1 = \operatorname{sgn}\beta_2$, the discriminant of Eq. (70) is strictly positive. Thus $\Psi \in \mathbb{R} \setminus \{0\}$ and $\operatorname{sgn} \Psi = \operatorname{sgn}\beta_1$. Let $\Xi \in \mathbb{R}$ be defined as

$$\Xi = \frac{1}{2}\left(\beta_1 - \beta_2^2\frac{1}{\Psi}\right).$$

Since $\alpha_1 \neq \alpha_2$ we see that $\Psi = \frac{\beta_2^2}{\beta_1}$ is not a solution to Eq (70) whence $\Xi \neq 0$. Let C_1, C_2 be as in the $\alpha_1 = \alpha_2$ case and let $\rho = \operatorname{sgn}\Xi$. Then Eq. (62) holds when g is the Lorentz metric given by Eq. (69) and $A = \frac{1}{2}A^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$ is given by

$$(A^{ij})_{ij} = \begin{pmatrix} 0 & \sqrt{|\Xi|} & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & \frac{\alpha_1-\alpha_2}{2\rho\sqrt{|\Xi|}} \\ & & & 0 \end{pmatrix}.$$

Metaclass II. If $\kappa|_p$ is in Metaclass II, there are coordinates $\{x^i\}_{i=0}^3$ around p such that κ is given by Eq. (32) with conditions on the parameters given by Theorem 3.4. Let $C_1 = -\frac{1}{\beta_1\sqrt{|\det g|}}$, $C_2 = \alpha_1$ and $\rho = 1/2$. Then Eq. (62) holds when $A = \frac{1}{2}A^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$ is as in Eq. (64) and g is the Lorentz metric $g = g_{ij}dx^i \otimes dx^j$ with coefficients

$$(g_{ij})_{ij} = \begin{pmatrix} -1 & 0 & 0 & \beta_1 \\ 0 & -\beta_1 & 0 & 0 \\ 0 & 0 & -\beta_1 & 0 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix}^{-1}. \tag{71}$$

Metaclass IV. If $\kappa|_p$ is in Metaclass IV, there are coordinates $\{x^i\}_{i=0}^3$ around p such that κ is given by Eq. (34) with conditions on the parameters given by Theorem 3.4. Suppose $\alpha_1 = \alpha_3$. Let $C_1 = \frac{\beta_1}{\Psi\sqrt{|\det g|}}$, $C_2 = \alpha_1$, $\Psi = \alpha_4/\beta_1$, and $\rho = (\alpha_4^2 + \beta_1^2)/(2\alpha_4)$. Then Eq. (62) holds when $A = \frac{1}{2}B^{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}$ when B^{ij} are as in Eq. (65) and g is the Lorentz metric $g = g_{ij}dx^i \otimes dx^j$ with coefficients

$$(g_{ij})_{ij} = (\mathrm{diag}(1, \Psi, \Psi, -1))^{-1}. \tag{72}$$

On the other hand, suppose $\alpha_1 \neq \alpha_3$. Let Ψ be one of the two roots to the quadratic equation

$$\Psi^2 + D_1\Psi - 1 = 0, \tag{73}$$

where (see Ref. 7, Theorem 2.1 (iii)),

$$D_1 = \frac{(\alpha_2-\alpha_3)^2 + \beta_2^2 - \alpha_4^2}{\beta_2\alpha_4}.$$

Then $\Psi \in \mathbb{R}\setminus\{0\}$ and since $\alpha_1 \neq \alpha_3$ Eq. (73) implies that $\Psi \neq \frac{\alpha_4}{\beta_1}$. Thus $\Xi \in \mathbb{R}\setminus\{0\}$ when

$$\Xi = \frac{1}{2}(\alpha_4 - \beta_1\Psi).$$

Let C_1, C_2 be as in the $\alpha_1 = \alpha_3$ case and let $\rho = \mathrm{sgn}\,\Xi$. Then Eq. (62) holds when g is the Lorentz metric in Eq. (72) and A is the bivector $A = \frac{1}{2}A^{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}$ with coefficients

$$(A^{ij})_{ij} = \begin{pmatrix} 0 & 0 & 0 & \frac{\alpha_3-\alpha_1}{2\rho\sqrt{|\Xi|}} \\ & 0 & \sqrt{|\Xi|} & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}.$$

This completes the proof of implication (i) ⇒ (iii). □

Let us first emphasise that the conditions in Theorem 5.1 are written analogously to the conditions in Theorem 3.2. In each theorem, condition (i) is the dynamical description of the medium, condition (ii) is a characterisation of the medium and condition *(iii)* is a general representation formula. Let us also emphasise that in suitable limits, condition (61) in Theorem 5.1 reduces to the closure condition $\kappa^2 = -\lambda\,\mathrm{Id}$ in Theorem 3.2, and representation formula (62) in Theorem 5.1 reduces to $\kappa = f{*_g}$ in Theorem 3.2. Let us also emphasise that in both theorems, all conditions are tensorial, and do not depend on coordinate expressions. A difference between the theorems is that Theorem 3.2 is a global result, while Theorem 5.1 is a pointwise result.

All the media in Theorem 5.1 satisfy the technical assumptions in Theorem 4.6 with either $D = A$ or $D = B$ when A and B are as in Eq. (61).

As described in the introduction, condition (ii) in Theorem 5.1 is a slight strengthening of the conditions derived in Ref. 24 (see Theorem 4.3 in the above). Representation formula (62) in Theorem 5.1 is also adapted from Ref. 24. For constant medium tensors on $\mathbb{R}^4$, Theorem 5.1 implies that if κ is invertible, skewon-free and decomposes into two Lorentz null cones, then κ is algebraically decomposable, and hence decomposable by Ref. 24 (see Theorem 4.3). In this setting, Theorem 5.1 explicitly shows that the behaviour of signal-speed imposes a constraint on the behaviour of polarisation. This can be seen as somewhat unexpected. However, the explanation is that polarisation and signal speeds are not independent for a propagating wave, but constrained by Eq. (51). For a further discussion, see Ref. 6. It is also instructive to note that condition (61) is a second order polynomial constraint on the coefficients in κ, but Definition 3.3 is a constraint involving third order polynomials of the coefficients in κ. The same phenomenon appears in equivalence (i) ⇔ (ii) in Theorem 3.2.

Part of condition (ii) is condition (a), that states that the Fresnel surface of κ contains no two dimensional subspace. Let us describe five results where this condition also appears. First, if the Fresnel surface of a $\kappa \in \widetilde{\Omega}^2{}_2(N)$ can be written as $F_p(\kappa) = \{\xi \in T_p^*(N) : (g(\xi,\xi))^2 = 0\}$ for a pseudo-Riemann metric g, then condition *(a)* is satisfied if and only if g has signature $(-\ -\ +\ +)$. This follows by a result of Montaldi.[27] For example, if $g = \mathrm{diag}(-1, -1, 1, 1)$, then $F_p(\kappa)$ contains the two-dimensional subspace $\mathrm{span}\{\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}\}$. Second, one can prove that condition *(a)* is always satisfied if the Fresnel surface of $\kappa|_p$ decomposes into two Lorentz null cones (see Proposition 1.3 in Ref. 7). Third, in matter dynamics systems, condition *(a)* can be motivated by the behaviour of energy.[35] In the terminology of Ref. 35, condition *(a)* can be replaced by the stronger condition that κ is *bihyperbolic*. Fourth, condition *(a)* also appears in the study of the well posedness of Maxwell's equations as an initial value problem.[38] Lastly, in the normal form representation of skewon-free medium tensors in Ref. 38, condition *(a)* simplifies the representation since the condition excludes all but the first 7 coordinate representations. See Ref. 38 and Sec. III A in the above.

When equivalence holds in Theorem 5.1, there does not seem to be a simple relation between parameters C_1, C_2, ρ, A, g in Eq. (62) and parameters μ, λ, ρ, A, B in Eq. (61). However, if Eq. (62) holds for an A such that $A \wedge A = 0$ (that is, A is *decomposable* or *simple* [Ref. 4, p. 185]), then Eq. (61) holds for parameters

$$\mu = -C_2, \quad \lambda = -C_1^2, \quad B = A(*_g).$$

Using a Gröbner basis argument one can show that the tensor κ defined by Eq. (31) when $\beta_1 = \beta_2 = \beta_3 = 1$, $\alpha_1 = 1$, and $\alpha_2 = \alpha_3 = 2$ is invertible and its Fresnel surface decompose into two Lorentz null cones. However, it cannot be written as in (Eq. (62) for an A such that $A \wedge A = 0$.

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[1] Bini, D. and Gemignani, L., "Solving quadratic matrix equations and factoring polynomials: New fixed point iterations based on Schur complements of Toeplitz matrices," Numer. Linear Algebra Appl. **12**, 181–189 (2005).
[2] Born, M. and Wolf, E., *Principles of Optics*, 7th ed. (Cambridge University Press, 1999).
[3] Choi, Y., Park, J., and Lee, S., "Closed-form solution of particular case of algebraic Riccati equation using Moser–Veselov equation," IET Control Theory Appl. **4**, 865–870 (2010).
[4] Cohn, P. M., *Basic Algebra: Groups, Rings, and Fields* (Springer, 2005).
[5] Cox, D., Little, J., and O'shea, D., *Ideals, Varieties, and Algorithms* (Springer, 2007).
[6] Dahl, M. F., "Determining electromagnetic medium from the Fresnel surface," J. Phys. A **45**, 405203 (2012).
[7] Dahl, M. F., "Non-dissipative electromagnetic medium with two Lorentz null cones," Ann. Phys. (in press).
[8] Dahl, M. F., "A restatement of the normal form theorem for area metrics," Int. J. Geom. Methods Mod. Phys. **9**, 1250046 (2012).
[9] Drummond, I. and Hathrell, S., "QED vacuum polarization in a background gravitational field and its effect on the velocity of photons," Phys. Rev. D **22**, 343–355 (1980).
[10] Favaro, A., "Recent advances in classical electromagnetic theory," Ph.D. tdissertation (Imperial College London, 2012).
[11] Favaro, A. and Bergamin, L., "The non-birefringent limit of all linear, skewonless media, and its unique light-cone structure," Ann. Phys. (Berlin) **523**, 383–401 (2011).
[12] Gohberg, I., Lancaster, P., and Rodman, L., *Indefinite Linear Algebra and Applications* (Birkhäuser, 2005).
[13] Greub, W., *Multilinear Algebra* (Springer, 1978).
[14] Grigoriev, D. and Schwarz, F., "Factoring and solving linear partial differential equations," Computing **73**, 179–197 (2004).
[15] Hehl, F. W., Obukhov, Y. N., and Rubilar, G. F., "Spacetime metric from linear electrodynamics II," e-print: arXiv:gr-qc/9911096.
[16] Hehl, F. and Obukhov, Y., *Foundations of Classical Electrodynamics: Charge, Flux, and Metric*, Progress in Mathematical Physics (Birkhäuser, 2003).
[17] Itin, Y., "Nonbirefringence conditions for spacetime," Phys. Rev. D **72**, 087502 (2005).
[18] Itin, Y., "On light propagation in premetric electrodynamics: the covariant dispersion relation," J. Phys. A **42**, 475402 (2009).
[19] Jadczyk, A., "Electromagnetic permeability and the vacuum and light-cone structure," Bull. Acad. Polon. Sci. (Phys. Astron.) **27**, 91–94 (1979).
[20] Lämmerzahl, C. and Hehl, F., "Riemannian light cone from vanishing birefringence in premetric vacuum electrodynamics," Phys. Rev. D **70**, 105022 (2004).
[21] Lerer, L. and Ran, A., "The discrete algebraic Riccati equation and Hermitian block Toeplitz matrices," In *A Panorama of Modern Operator Theory and Related Topics*, Operator Theory: Advances and Applications Vol. 218, edited by H. Dym, M. Kaashoek, P. Lancaster, H. Langer, and L. Lerer, (Springer, 2012), pp. 495–512.

[22] Li, T., Zhang, Z., and Wang, T., "Determining the structure of the Jordan normal form of a matrix by symbolic computation," Linear Algebra Appl. **252**, 221–259 (1997).
[23] Lindell, I., "Class of electromagnetic SD media," Metamaterials **2**, 54–70 (2008).
[24] Lindell, I., Bergamin, L., and Favaro, A., "Decomposable medium conditions in four-dimensional representation," IEEE Trans. Antennas Propag. **60**, 367–376 (2012).
[25] Lindell, I. and Wallén, K., "Wave equations for bi-anisotropic media in differential forms," J. Electromagn. Waves Appl. **16**, 1615–1635 (2002).
[26] Lindell, I. and Wallén, K., "Differential-form electromagnetics and bi-anisotropic Q-media," J. Electromagn. Waves Appl. **18**, 957–968 (2004).
[27] Montaldi, J., "A note on the geometry of linear Hamiltonian systems of signature 0 in $\mathbb{R}^4$," Diff. Geom. Applic. **25**, 344–350 (2007).
[28] Obukhov, Y., Fukui, T., and Rubilar, G., "Wave propagation in linear electrodynamics," Phys. Rev. D **62**, 044050 (2000).
[29] Obukhov, Y. and Hehl, F., "Spacetime metric from linear electrodynamics," Phys. Lett. B **458**, 466–470 (1999).
[30] Obukhov, Y. and Hehl, F., "Possible skewon effects on light propagation," Phys. Rev. D **70**, 125015 (2004).
[31] Obukhov, Y. and Rubilar, G., "Fresnel analysis of wave propagation in nonlinear electrodynamics," Phys. Rev. D **66**, 024042 (2002).
[32] Perlick, V., "On the hyperbolicity of Maxwell's equations with a local constitutive law" J. Math. Phys. **52** 042903 (2011).
[33] Punzi, R., Schuller, F., and Wohlfarth, M., "Area metric gravity and accelerating cosmology," J. High Energy Phys. **02**, 030 (2007).
[34] Punzi, R., Schuller, F., and Wohlfarth, M., "Propagation of light in area metric backgrounds," Class. Quantum Grav. **26**, 035024 (2009).
[35] Rätzel, D., Rivera, S., and Schuller, F., "Geometry of physical dispersion relations," Phys. Rev. D **83**, 044047 (2011).
[36] Rubilar, G., "Linear pre-metric electrodynamics and deduction of the light cone," Ann. Phys. (Berlin) **11**, 717–782 (2002).
[37] Schönberg, M., "Electromagnetism and gravitation," Rivista Brasileira de Fisica **1**, 91–122 (1971).
[38] Schuller, F., Witte, C., and Wohlfarth, M., "Causal structure and algebraic classification of non-dissipative linear optical media," Ann. Phys. **325**, 1853–1883 (2010).
[39] Spivak, M., *A Comprehensive Introduction to Differential Geometry*, 3rd ed. (Publish or Perish, Inc., 1999), Vol. 1.
[40] Toupin, R., "Elasticity and electro-magnetism," Nonlinear Continuum Theor. **36**, 203–342 (2011), Reissue of article from C.I.M.E. Conference in Bressanone, Italy (1965).