A RESTATEMENT OF THE NORMAL FORM THEOREM FOR AREA METRICS

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Abstract. An *area metric* is a $\binom{0}{4}$-tensor with certain symmetries on a 4-manifold that represent a non-dissipative linear electromagnetic medium. A recent result by Schuller, Witte and Wohlfarth provides a pointwise normal form theorem for such area metrics. This result is similar to the Jordan normal form theorem for $\binom{1}{1}$-tensors, and the result shows that any area metric belongs to one of 23 metaclasses with explicit coordinate expressions for each metaclass. In this paper we restate and prove this result for skewon-free $\binom{2}{2}$-tensors and show that in general, each metaclasses has three different coordinate representations, and each of metaclasses I, II, . . ., VI, VII need only one coordinate representation.

1. Introduction

An *area metric* on a 4-manifold N is a $\binom{0}{4}$-tensor G on N that gives a symmetric (possibly indefinite) inner product for bivectors on N. The motivation for studying area metrics is that they appear as a natural generalisation of Lorentz metrics in physics. For example, in relativistic electromagnetics, a Lorentz metric always describes an isotropic medium, but using an area metric one can also model anisotropic medium, where differently polarised waves can propagate with different wave-speeds. Area metrics also appear when studying the propagation of a photon in a vacuum with a first order correction from quantum electrodynamics [DH80, SWW10]. The Einstein field equations have also been generalised into equations where the unknown field is an area metric [PSW07]. For further examples, see [PSW09, SWW10], and for the differential geometry of area metrics, see [SW06, PSW07].

The present work is motivated by a recent result by Schuller, Witte and Wohlfarth [SWW10] which is a normal form theorem for area metrics on a 4-manifold N. Essentially, this theorem states that there are 23 normal forms for area metrics, and if G is any area metric on N and $p \in N$, one can find coordinates around p such that $G|_p$ is one of the normal forms (up to simple operations) [SWW10, Theorem 4.3]. What is more, 16 of the metaclasses are unphysical in the sense that Maxwell's equations are not well-posed in these metaclasses. This leaves only 7 metaclasses that can describe physically relevant electromagnetic medium [SWW10]. The importance of this result is that in arbitrary coordinates an area metric depends on 21 real numbers, but each normal form depend on at most 6 real numbers and 3 signs ± 1. This reduction in variables has proven particularly useful when studying properties of the Fresnel equation (or dispersion equation) for a propagating electromagnetic wave [SWW10, FB11]. Namely, without assumptions on either the area metric or the coordinates, the Fresnel equation usually leads to algebraic expressions that are quite difficult to manipulate, even with computer algebra [Dah11].

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In addition to area metrics, there are multiple other ways to model the medium in (relativistic) electrodynamics. Another common formalism is the so called *pre-metric* formulation, where the medium is modelled by an antisymmetric $\binom{2}{2}$-tensor κ on a 4-manifold N. In this formalism, an electromagnetic medium κ is pointwise determined by 36 real numbers [HO03]. Under suitable conditions it follows that the area-metrics on N are in one-to-one correspondence with invertible skewon-free $\binom{2}{2}$-tensors on N. (See [FB11] and Propositions 2.1 and 2.5 below). Because of this correspondence, the normal form theorem in [SWW10] can, of course, be stated also for skewon-free $\binom{2}{2}$-tensors. The contribution of this paper we write down this restatement explicitly, and also prove the result in this setting by following the proof in [SWW10]. Below, this is given by Theorem 3.2. However, we obtain a slightly different result. In [SWW10], area metrics divide into 23 metaclasses and each metaclass has two representations in local coordinates, but in Theorem 3.2, we obtain three different coordinate representations for each metaclass. Moreover, for metaclasses I, II, ..., VI, VII we show that only one coordinate representation is needed per metaclass.

A minor difference is also that in Theorem 3.2, one does not need to assume that κ is invertible. This was already noted in [FB11].

This paper relies on computations by computer algebra. Mathematica notebooks for these computations can be found on the author's homepage.

2. Maxwell's equations

By a *manifold* M we mean a second countable topological Hausdorff space that is locally homeomorphic to $\mathbb{R}^n$ with C^∞-smooth transition maps. All objects are assumed to be smooth and real where defined. Let TM and T^*M be the tangent and cotangent bundles, respectively, and for $k \geq 1$, let $\Lambda^k(M)$ be the set of antisymmetric k-covectors, so that $\Lambda^1(N) = T^*N$. Also, let $\Lambda_k(M)$ be the set of antisymmetric k-vectors. Let $\Omega^k_l(M)$ be $\binom{k}{l}$-tensors that are antisymmetric in their k upper indices and l lower indices. In particular, let $\Omega^k(M)$ be the set of k-forms. Let $C^\infty(M)$ be the set of functions. The Einstein summing convention is used throughout. When writing tensors in local coordinates we assume that the components satisfy the same symmetries as the tensor.

2.1. Maxwell's equations on a 4-manifold.

Suppose N is a 4-manifold. On a 4-manifold N, Maxwell's equations read

$$dF = 0, \tag{1}$$

$$dG = j, \tag{2}$$

where d is the exterior derivative on N, $F, G \in \Omega^2(N)$, and $j \in \Omega^3(N)$. By an *electromagnetic medium* on N we mean a map

$$\kappa\colon \Omega^2(N) \to \Omega^2(N).$$

We then say that 2-forms $F, G \in \Omega^2(N)$ *solve Maxwell's equations in medium* κ if F and G satisfy equations (1)–(2) and

$$G = \kappa(F). \tag{3}$$

Equation (3) is known as the *constitutive equation*. If κ is invertible, it follows that one can eliminate half of the free variables in Maxwell's equations (1)–(2). We assume that κ is linear and determined pointwise so that we can represent κ by an antisymmetric $\binom{2}{2}$-tensor $\kappa \in \Omega^2_2(N)$. If in coordinates $\{x^i\}_{i=0}^3$ for N we have

$$\kappa \quad = \quad \frac{1}{2}\kappa^{ij}_{lm} dx^l \otimes dx^m \otimes \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \tag{4}$$

and $F = F_{ij} dx^i \otimes dx^j$ and $G = G_{ij} dx^i \otimes dx^j$, then constitutive equation (3) reads

$$G_{ij} \quad = \quad \frac{1}{2}\kappa^{rs}_{ij} F_{rs}. \tag{5}$$

2.2. **Decomposition of electromagnetic medium.** Let N be a 4-manifold. Then at each point on N, a general antisymmetric $\binom{2}{2}$-tensor depends on 36 parameters. Such tensors canonically decompose into three linear subspaces. The motivation for this decomposition is that different components in the decomposition enter in different parts of electromagnetics. See [HO03, Section D.1.3]. The below formulation is taken from [Dah09].

If $\kappa \in \Omega^2_2(N)$ we define the *trace* of κ as the smooth function $N \to \mathbb{R}$ given by $\operatorname{trace} \kappa = \frac{1}{2}\kappa^{ij}_{ij}$ when κ is locally given by equation (4). Writing Id as in equation (4) gives $\mathrm{Id}^{ij}_{rs} = \delta^i_r \delta^j_s - \delta^i_s \delta^j_r$, so $\operatorname{trace} \mathrm{Id} = 6$ when $\dim N = 4$.

Proposition 2.1 (Decomposition of a $\binom{2}{2}$-tensors)**.** *Let N be a 4-manifold, and let*

$$\begin{aligned}
Z &= \{\kappa \in \Omega^2_2(N) : u \wedge \kappa(v) = \kappa(u) \wedge v \textit{ for all } u, v \in \Omega^2(N), \\
&\qquad\qquad \operatorname{trace} \kappa = 0\}, \\
W &= \{\kappa \in \Omega^2_2(N) : u \wedge \kappa(v) = -\kappa(u) \wedge v \textit{ for all } u, v \in \Omega^2(N)\} \\
&= \{\kappa \in \Omega^2_2(N) : u \wedge \kappa(v) = -\kappa(u) \wedge v \textit{ for all } u, v \in \Omega^2(N), \\
&\qquad\qquad \operatorname{trace} \kappa = 0\}, \\
U &= \{f\,\mathrm{Id} \in \Omega^2_2(N) : f \in C^\infty(N)\}.
\end{aligned}$$

Then

$$\Omega^2_2(N) \quad = \quad Z \,\oplus\, W \,\oplus\, U, \tag{6}$$

and pointwise, $\dim Z = 20$*,* $\dim W = 15$ *and* $\dim U = 1$*.*

If we write a $\kappa \in \Omega^2_2(N)$ as

$$\kappa \quad = \quad {}^{(1)}\kappa \,+\, {}^{(2)}\kappa \,+\, {}^{(3)}\kappa$$

with ${}^{(1)}\kappa \in Z$, ${}^{(2)}\kappa \in W$, ${}^{(3)}\kappa \in U$, then we say that ${}^{(1)}\kappa$ is the *principal part*, ${}^{(2)}\kappa$ is the *skewon part*, ${}^{(3)}\kappa$ is the *axion part* of κ.

2.3. **Representing κ as a 6×6 matrix.** Let O be the ordered set of index pairs $\{01, 02, 03, 23, 31, 12\}$. If $I \in O$, let us also denote the individual indices by I_1 and I_2. Say, if $I = 31$ then $I_2 = 1$.

If $\{x^i\}_{i=0}^3$ are local coordinates for a 4-manifold N, and $J \in O$ we define $dx^J = dx^{J_1} \wedge dx^{J_2}$. A basis for $\Omega^2(N)$ is given by $\{dx^J : J \in O\}$, that is,

$$\{dx^0 \wedge dx^1, dx^0 \wedge dx^2, dx^0 \wedge dx^3, dx^2 \wedge dx^3, dx^3 \wedge dx^1, dx^1 \wedge dx^2\}. \tag{7}$$

This choice of basis follows [HO03, Section A.1.10] and [FB11].

If $\kappa \in \Omega^2_2(N)$ is written as in equation (4) and $J \in O$, then

$$\kappa(dx^J) = \sum_{I \in O} \kappa^J_I dx^I = \kappa^J_I dx^I, \quad J \in O, \tag{8}$$

where $\kappa_I^J = \kappa_{I_1 I_2}^{J_1 J_2}$ and in the last equality we have extended the Einstein summing convention also to elements in O. We will always use capital letters $I, J, K, \ldots$ to denote elements in O.

Let b be the natural bijection $b\colon O \to \{1, \ldots, 6\}$. Then coefficients $\{\kappa_I^J : I, J \in O\}$ can be identified with a 6×6 matrix. To do this identification in a systematic way, we denote by $(m(I,J))_{IJ}$ the 6×6 matrix whose entry at row $b(I) \in \{1, \ldots, 6\}$ and column $b(J) \in \{1, \ldots, 6\}$ is given by expression $m(I,J)$. For example, if A is the 6×6 matrix $A = (\kappa_I^J)_{IJ}$, then

$$\kappa_I^J \quad = \quad A_{b(I)b(J)}, \quad I, J \in O. \tag{9}$$

If $\eta \in \Omega_2^2(N)$ and $B = (\eta_I^J)_{IJ}$, where η_I^J represent $\eta|_p$ as in equation (8), then $((\kappa \circ \eta)_I^J)_{IJ} = AB$. This compatibility with the matrix multiplication is the motivation for using the matrix representation for κ as in equation (9) (and not the transpose of A).

Suppose $\{x^i\}_{i=0}^3$ and $\{\widetilde{x}^i\}_{i=0}^3$ are overlapping coordinates, and suppose that in these coordinates κ is represented by κ_I^J and $\widetilde{\kappa}_I^J$ as in equation (8). Then we have the transformation rule

$$\widetilde{\kappa}_I^J \quad = \quad \frac{\partial \widetilde{x}^J}{\partial x^K} \kappa_L^K \frac{\partial x^L}{\partial \widetilde{x}^I}, \quad I, J \in O, \tag{10}$$

where

$$\frac{\partial \widetilde{x}^I}{\partial x^J} \quad = \quad \frac{\partial \widetilde{x}^{I_1}}{\partial x^{J_1}} \frac{\partial \widetilde{x}^{I_2}}{\partial x^{J_2}} - \frac{\partial \widetilde{x}^{I_2}}{\partial x^{J_1}} \frac{\partial \widetilde{x}^{I_1}}{\partial x^{J_2}}, \quad I, J \in O$$

and $\frac{\partial x^I}{\partial \widetilde{x}^J}$ is defined analogously by exchanging x and $\widetilde{x}$. For $I, J \in O$, we then have $\frac{\partial \widetilde{x}^I}{\partial x^K} \frac{\partial x^K}{\partial \widetilde{x}^J} = \delta_J^I$, where $\delta_J^I = \delta_{J_1}^{I_1} \delta_{J_2}^{I_2} - \delta_{J_1}^{I_2} \delta_{J_2}^{I_1}$.

2.4. **The Hodge star operator.** By a *pseudo-Riemann metric* on a manifold M we mean a symmetric $\binom{0}{2}$-tensor g that is non-degenerate. If M is not connected we also assume that g has constant signature. If g is positive definite we say that g is a *Riemann metric.*

Suppose g is a pseudo-Riemann metric on an orientable manifold M with $n = \dim M \geq 1$. For $p \in \{0, \ldots, n\}$, the *Hodge star operator* $*$ is the linear map $*\colon \Omega^p(M) \to \Omega^{n-p}(M)$ defined as [AMR88, p. 413]

$$*(dx^{i_1} \wedge \cdots \wedge dx^{i_p}) \quad = \quad \frac{\sqrt{|\det g|}}{(n-p)!} g^{i_1 l_1} \cdots g^{i_p l_p} \varepsilon_{l_1 \cdots l_p \, l_{p+1} \cdots l_n} dx^{l_{p+1}} \wedge \cdots \wedge dx^{l_n},$$

where x^i are local coordinates in an oriented atlas, $g = g_{ij} dx^i \otimes dx^j$, $\det g = \det g_{ij}$, g^{ij} is the ijth entry of $(g_{ij})^{-1}$, and $\varepsilon_{l_1 \cdots l_n}$ is the *Levi-Civita permutation symbol.* We treat $\varepsilon_{l_1 \cdots l_n}$ as a purely combinatorial object (and not as a tensor density). We also define $\varepsilon^{l_1 \cdots l_n} = \varepsilon_{l_1 \cdots l_n}$.

If g is a pseudo-Riemann metric on an oriented 4-manifold N, then the Hodge star operator for g induces a $\binom{2}{2}$-tensor $\kappa = *_g \in \Omega_2^2(N)$. If κ is written as in equation (4) for local coordinates x^i then

$$\kappa_{rs}^{ij} \quad = \quad \sqrt{|g|} g^{ia} g^{jb} \varepsilon_{abrs} \tag{11}$$

and κ has only a principal part. See for example [Dah11, Proposition 2.2].

Next we show that two pseudo-Riemann metrics can be combined by conjugation into a third pseudo-Riemann metric.

Proposition 2.2. *Suppose g and h are pseudo-Riemann metrics on an orientable 4-dimensional manifold N. Then the pseudo-Riemann metric k defined as*

$$k = g_{ia}h^{ab}g_{bj}dx^i \otimes dx^j$$

satisfies

$$*_k = \operatorname{sgn}\left(\frac{\det g}{\det h}\right) *_g^{-1} \circ *_h \circ *_g .$$

Conversely, if $\widetilde{k}$ is a pseudo-Riemann metric such that $*_{\widetilde{k}} = \lambda *_g^{-1} \circ *_h \circ *_g$ for some $\lambda \in C^\infty(N)$, then k and $\widetilde{k}$ are in the same conformal class.

Proof. Let g_{ij}, h_{ij} and k_{ij} be components for g, h and k, respectively. Using $\varepsilon_{ijkl}A^{ia}A^{jb}A^{kc}A^{ld} = \varepsilon^{abcd}\det A$ we obtain obtain

$$*_k(dx^i \wedge dx^j) = \operatorname{sgn}(\det g)\frac{|\det h|^{-1/2}}{2}g^{ia}g^{jb}h_{ac}h_{bd}\varepsilon^{cdrs}g_{ru}g_{sv}dx^u \wedge dx^v$$

Similarly writing out $*_g^{-1} \circ *_h \circ *_g$ gives the first claim. The second claim follows by the lemma below. □

The next lemma is a slight generalisation of Theorem 1 in [DKS89].

Lemma 2.3. *Suppose g and h are pseudo-Riemann metrics on an orientable 4-dimensional manifold N. If $*_g = f *_h$ for some $f \in C^\infty(N)$, then $f = 1$ and $g = \lambda h$ for some $\lambda \in C^\infty(N)$.*

Proof. Since we only need to prove the claim at one point, let $\{x^i\}_{i=0}^3$ be coordinates for a connected neighbourhood U around some $p \in N$ where $h|_p$ is diagonal with entries ± 1. Squaring $*_g = f *_h$ gives $f^2 = 1$, so in U we have either $f = 1$ or $f = -1$. By equation (11),

$$\sqrt{|\det g|}g^{ia}g^{jb}\varepsilon_{abrs} = f\sqrt{|\det h|}h^{ia}h^{jb}\varepsilon_{abrs}. \tag{12}$$

Contracting by ε^{mnrs} and using equation (17) gives

$$\sqrt{|\det g|}\left(g^{ij}g^{kl} - g^{ik}g^{jl}\right) = f\sqrt{|\det h|}\left(h^{ij}h^{kl} - h^{ik}h^{jl}\right) \tag{13}$$

for all $i, j, k, l \in \{0, 1, 2, 3\}$. Thus, if we have neither $[i = j$ and $k = l]$ nor $[i = k$ and $j = l]$, then

$$g^{ij}g^{kl} = g^{ik}g^{jl}. \tag{14}$$

Thus, if i, j, k, l are distinct, then

$$\begin{aligned}\left(g^{ii}g^{jj} - (g^{ij})^2\right)(g^{kl})^2 &= \left(g^{ii}g^{kl}\right)\left(g^{jj}g^{kl}\right) - \left(g^{ij}g^{kl}\right)\left(g^{ij}g^{lk}\right)\\ &= \left(g^{ik}g^{il}\right)\left(g^{jk}g^{jl}\right) - \left(g^{ik}g^{jl}\right)\left(g^{il}g^{jk}\right)\\ &= 0,\end{aligned} \tag{15}$$

Combining equations (13) and (15) gives $(h^{ii}h^{jj} - (h^{ij})^2)(g^{kl})^2 = 0$. Hence g is also diagonal at p. Equation (13) gives

$$\sqrt{|\det g|}g^{ii}g^{jj} = f\sqrt{|\det h|}h^{ii}h^{jj}, \quad i < j. \tag{16}$$

Writing out equation (16) for cases $(i, j) = (0, 3), (1, 3)$ and $(0, 1)$ and multiplying the first two equations gives $\sqrt{|\det g|}(g^{33})^2 = f\sqrt{|\det h|}(h^{33})^2$. Thus $f = 1$ in U and $g^{33} = \sigma h^{33}$ for some $\sigma \in \{\pm\left(\frac{|\det h|}{|\det g|}\right)^{1/4}\}$. Setting $j = 3$ in equation (16) then gives $g^{ii} = \sigma h^{ii}$ for $i \in \{0, 1, 2\}$. □

2.5. **Area metrics.** As described in the introduction, an area metric is a geometry that at each point p gives a (possible indefinite) inner product for bivectors, that is, for elements in $\Lambda_2(N)|_p$. In this section we show that area metrics are essentially in one-to-one correspondence with skewon-free $\binom{2}{2}$-tensors.

Definition 2.4. Suppose N is a 4-manifold. An *area metric* is a $\binom{0}{4}$-tensor G on N such that

(i) $G(u, v, p, q)$ is antisymmetric in u, v,
(ii) For each $p \in N$, the quadratic form
$$\Lambda_2(N)|_p \times \Lambda_2(N)|_p \quad \to \quad \mathbb{R}$$
determined by
$$(a \wedge b, u \wedge v) \quad \mapsto \quad G(a, b, u, v), \quad a, b, u, v \in \Lambda^1_p(N)$$
is symmetric and non-degenerate.

Suppose G is a $\binom{0}{4}$-tensor on a 4-manifold. Then in local coordinates $\{x^i\}_{i=0}^3$,
$$G = G_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l,$$
and G is an area metric if and only if components G_{ijkl} satisfy *(i)* $G_{ijrs} = -G_{jirs}$, *(ii)* $G_{ijrs} = G_{rsij}$ and *(iii)* the 6×6 matrix $(G_{I_1 I_2 J_1 J_2})_{IJ}$ is invertible.

Proposition 2.5. *Suppose N is an orientable 4-manifold and g is a pseudo-Riemann metric on N. Let A be the map that maps a $\kappa \in \Omega^2_2(N)$ into the $\binom{0}{4}$-tensor*
$$A(\kappa) \quad = \quad \frac{1}{2} |\det g|^{1/2} \kappa^{rs}_{ij} \varepsilon_{rskl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l,$$
where κ^{ij}_{kl} are defined as in equation (4). Then $\kappa \mapsto A(\kappa)$ is invertible, and if κ is invertible as a linear map $\kappa \colon \Omega^2(N) \to \Omega^2(N)$, then κ is skewon-free if and only if $A(\kappa)$ is an area metric.

Proof. A direct computation shows that $A(\kappa)$ is a tensor, and the identity
$$\varepsilon^{ijkl} \varepsilon_{ijrs} = 2(\delta^k_r \delta^l_s - \delta^k_s \delta^l_r) \tag{17}$$
shows that A is invertible. Using equation (17) we also see that for the last equivalence we only need to show that $(\kappa^{rs}_{I_1 I_2} \varepsilon_{rs J_1 J_2})_{IJ}$ is an invertible 6×6 matrix. The result follows since the summation over r, s can be written as a matrix multiplication. □

3. The normal form theorem restated for $\binom{2}{2}$-tensors

In this section we formulate Theorem 3.2, which provides the restatement of the normal form theorem in [SWW10]. First we introduce some terminology and notation. Suppose $L \colon V \to V$ is a linear map where V is a real n-dimensional vector space. If the matrix representation of L in some basis is $A \in \mathbb{R}^{n \times n}$ and A is written as in Theorem B.1, then we say that L has *Segre type* $\left[m_1 \cdots m_r \, k_1 \overline{k_1} \cdots k_s \overline{k_s}\right]$. Moreover, by Theorem B.1, the Segre type depends only on L and not on the basis. If $\kappa \in \Omega^2_2(N)|_p$, where N is a 4-manifold and $p \in N$, then we say that the Segre type of $\kappa|_p$ is the Segre type of the linear map $\kappa|_p \colon \Omega^2(N)|_p \to \Omega^2_p(N)|_p$. By counting how many ways a 6×6 matrix can be partitioned into blocks as in equation (51), we see that there are 23 possible Segre types for $\kappa|_p$. These are the Segre types listed in Theorem 3.2, that is,
$$[1\overline{1}\,1\overline{1}\,1\overline{1}], \quad [2\overline{2}\,1\overline{1}], \quad [3\overline{3}], \quad \cdots \quad [321], \quad [31\,1\overline{1}], \quad [31\,11]. \tag{18}$$

To formulate Theorem 3.2 we need the following definition.

Definition 3.1. Suppose N is a 4-dimensional manifold, $\kappa \in \Omega^2_2(N)$, $p \in N$ and $V \in \mathbb{R}^{6\times 6}$. We then write

$$\kappa|_p \quad \sim \quad V \tag{19}$$

to indicate that there exist coordinates $\{x^i\}_{i=0}^3$ around p for which at least one of the below conditions is satisfied:

(i) $(\kappa_I^J)_{IJ} = V$.

(ii) For the Riemann metric $g = \mathrm{diag}(1,1,1,1)$ in coordinates $\{x^i\}_{i=0}^3$ we have
$$((*_g \circ \kappa \circ *_g)_I^J)_{IJ} \quad = \quad V.$$

(iii) For the pseudo-Riemann metric $g = \mathrm{diag}(1,-1,-1,1)$ in coordinates $\{x^i\}_{i=0}^3$ we have
$$((*_g \circ \kappa \circ *_g)_I^J)_{IJ} \quad = \quad V.$$

In the above η_I^J denote the components as in equation (8) that determine $\eta|_p$ in coordinates $\{x^i\}_{i=0}^3$ when $\eta \in \Omega^2_2(N)$.

Let us make three remarks regarding Definition 3.1. First, for the metrics in conditions *(ii)* and *(iii)* we have $*_g = *_g^{-1}$, so the operations in *(ii)* and *(iii)* are just conjugation by a Hodge star operator. Proposition 2.2 shows that this is a natural operator in the sense that in the class of pseudo-Riemann metrics, the operation is closed (up to a sign depending on signature). Second, if $g = \mathrm{diag}(1,1,1,1)$ and if we use the correspondence in Proposition 2.5, then conjugation $\kappa \mapsto *_g^{-1} \circ \kappa \circ *_g$ for invertible skewon-free $\binom{2}{2}$-tensors corresponds to conjugation $G \mapsto \Sigma^t \cdot G \cdot \Sigma$ for area metrics in [SWW10, Theorem 4.10]. Second, Proposition A.1 in Appendix A gives two alternative descriptions for the property $\kappa|_p \sim V$. Third, conditions *(i)*, *(ii)* and *(iii)* are not mutually exclusive. If $\kappa = \mathrm{Id}$ then all conditions are equivalent.

Theorem 3.2. *Let $\kappa \in \Omega^2_2(N)$, and suppose that ${}^{(2)}\kappa|_p = 0$ for some $p \in N$. Then $\kappa|_p \sim V$ for a matrix $V \in \mathbb{R}^{6\times 6}$ from the below list of matrices (listed with Segre type). Moreover, if $\kappa|_p$ has Segre type $I, II, \dots, VI, VII$, then we may assume that $\kappa|_p \sim V$ holds with alternative* (i) *in Definition 3.1.*

- *Metaclass I:* $[1\bar{1}\,1\bar{1}\,1\bar{1}]$

$$\begin{pmatrix} \alpha_1 & 0 & 0 & -\beta_1 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & -\beta_2 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & -\beta_3 \\ \beta_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \beta_3 & 0 & 0 & \alpha_3 \end{pmatrix}$$

- *Metaclass II:* $[2\bar{2}\,1\bar{1}]$

$$\begin{pmatrix} \alpha_1 & -\beta_1 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & -\beta_2 \\ 0 & 1 & 0 & \alpha_1 & \beta_1 & 0 \\ 1 & 0 & 0 & -\beta_1 & \alpha_1 & 0 \\ 0 & 0 & \beta_2 & 0 & 0 & \alpha_2 \end{pmatrix}$$

- *Metaclass III:* $[3\overline{3}]$

$$\begin{pmatrix} \alpha_1 & -\beta_1 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 1 & 0 & \alpha_1 & 0 & 0 & -\beta_1 \\ 0 & 0 & 0 & \alpha_1 & \beta_1 & 1 \\ 0 & 0 & 1 & -\beta_1 & \alpha_1 & 0 \\ 0 & 1 & \beta_1 & 0 & 0 & \alpha_1 \end{pmatrix}$$

- *Metaclass IV:* $[11\,1\overline{1}\,1\overline{1}]$

$$\begin{pmatrix} \alpha_1 & 0 & 0 & -\beta_1 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & -\beta_2 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_4 \\ \beta_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_4 & 0 & 0 & \alpha_3 \end{pmatrix}$$

- *Metaclass V:* $[11\,2\overline{2}]$

$$\begin{pmatrix} \alpha_1 & -\beta_1 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & \alpha_3 \\ 0 & 1 & 0 & \alpha_1 & \beta_1 & 0 \\ 1 & 0 & 0 & -\beta_1 & \alpha_1 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_2 \end{pmatrix}$$

- *Metaclass VI:* $[11\,11\,1\overline{1}]$

$$\begin{pmatrix} \alpha_1 & 0 & 0 & -\beta_1 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & \alpha_4 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_5 \\ \beta_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \alpha_4 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_5 & 0 & 0 & \alpha_3 \end{pmatrix}$$

- *Metaclass VII:* $[11\,11\,11]$

$$\begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & \alpha_5 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_6 \\ \alpha_4 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \alpha_5 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_6 & 0 & 0 & \alpha_3 \end{pmatrix}$$

- *Metaclass VIII:* $[6]$ $\quad \epsilon_1 \in \{\pm 1\}$

$$\begin{pmatrix} \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \alpha_1 & 1 \\ 0 & 0 & \epsilon_1 & 0 & 0 & \alpha_1 \end{pmatrix}$$

- *Metaclass IX:* [42] $\epsilon_1, \epsilon_2 \in \{\pm 1\}$

$$\begin{pmatrix} \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 & 1 & 0 \\ 0 & \epsilon_1 & 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & \epsilon_2 & 0 & 0 & \alpha_2 \end{pmatrix}$$

- *Metaclass X:* $[4\,1\bar{1}]$ $\epsilon_1 \in \{\pm 1\}$

$$\begin{pmatrix} \alpha_2 & 0 & 0 & 0 & 0 & 0 \\ 1 & \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 & 0 & -\beta_1 \\ 0 & 0 & 0 & \alpha_2 & 1 & 0 \\ 0 & \epsilon_1 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \beta_1 & 0 & 0 & \alpha_1 \end{pmatrix}$$

- *Metaclass XI:* [411] $\epsilon_1 \in \{\pm 1\}$

$$\begin{pmatrix} \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & \alpha_3 \\ 0 & 0 & 0 & \alpha_1 & 1 & 0 \\ 0 & \epsilon_1 & 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_2 \end{pmatrix}$$

- *Metaclass XII:* $[2\,2\bar{2}]$ $\epsilon_1 \in \{\pm 1\}$

$$\begin{pmatrix} \alpha_1 & -\beta_1 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_1 & \beta_1 & 0 \\ 1 & 0 & 0 & -\beta_1 & \alpha_1 & 0 \\ 0 & 0 & \epsilon_1 & 0 & 0 & \alpha_2 \end{pmatrix}$$

- *Metaclass XIII:* [222] $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}, \epsilon_1 \leq \epsilon_2 \leq \epsilon_3$

$$\begin{pmatrix} \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & \epsilon_2 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & 0 \\ \epsilon_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \epsilon_3 & 0 & 0 & \alpha_3 \end{pmatrix}$$

- *Metaclass XIV:* $[22\,1\bar{1}]$ $\epsilon_1, \epsilon_2 \in \{\pm 1\}, \epsilon_1 \leq \epsilon_2$

$$\begin{pmatrix} \alpha_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & \epsilon_2 & 0 \\ 0 & 0 & \alpha_1 & 0 & 0 & -\beta_1 \\ \epsilon_1 & 0 & 0 & \alpha_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \beta_1 & 0 & 0 & \alpha_1 \end{pmatrix}$$

- *Metaclass XV:* $[22\,11]$ $\quad \epsilon_1, \epsilon_2 \in \{\pm 1\}, \epsilon_1 \leq \epsilon_2$

$$\begin{pmatrix} \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & \epsilon_1 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_4 \\ \epsilon_2 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_4 & 0 & 0 & \alpha_3 \end{pmatrix}$$

- *Metaclass XVI:* $[2\,1\bar{1}\,1\bar{1}]$ $\quad \epsilon_1 \in \{\pm 1\}$

$$\begin{pmatrix} \alpha_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 & -\beta_1 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & -\beta_2 \\ \epsilon_1 & 0 & 0 & \alpha_3 & 0 & 0 \\ 0 & \beta_1 & 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & \beta_2 & 0 & 0 & \alpha_2 \end{pmatrix}$$

- *Metaclass XVII:* $[211\,1\bar{1}]$ $\quad \epsilon_1 \in \{\pm 1\}$

$$\begin{pmatrix} \alpha_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 & -\beta_1 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_4 \\ \epsilon_1 & 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & \beta_1 & 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_4 & 0 & 0 & \alpha_3 \end{pmatrix}$$

- *Metaclass XVIII:* $[21111]$ $\quad \epsilon_1 \in \{\pm 1\}$

$$\begin{pmatrix} \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & \alpha_4 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_5 \\ \epsilon_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \alpha_4 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_5 & 0 & 0 & \alpha_3 \end{pmatrix}$$

- *Metaclass XIX:* $[51]$ $\quad \epsilon_1 \in \{\pm 1\}$

$$\begin{pmatrix} \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2}(\alpha_1+\alpha_2) & 0 & 0 & \frac{\epsilon_1}{2}(\alpha_1-\alpha_2) \\ 0 & 0 & 0 & \alpha_1 & 1 & 0 \\ 0 & 0 & \frac{\epsilon_1}{\sqrt{2}} & 0 & \alpha_1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{\epsilon_1}{\sqrt{2}} & \frac{\epsilon_1}{2}(\alpha_1-\alpha_2) & 0 & 0 & \frac{1}{2}(\alpha_1+\alpha_2) \end{pmatrix}$$

- *Metaclass XX:* $[33]$

$$\begin{pmatrix} \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{2}(\alpha_1+\alpha_2) & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{2}(\alpha_1-\alpha_2) & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \alpha_1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{2}(\alpha_1-\alpha_2) & \frac{1}{\sqrt{2}} & 0 & \frac{1}{2}(\alpha_1+\alpha_2) & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & \alpha_2 \end{pmatrix}$$

- *Metaclass XXI:* [321] $\quad \epsilon_1, \epsilon_2 \in \{\pm 1\}$

$$\begin{pmatrix}
\alpha_1 & 0 & 0 & \epsilon_2 & 0 & 0 \\
0 & \alpha_2 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{\epsilon_1}{\sqrt{2}} \\
0 & 0 & \frac{1}{2}(\alpha_2+\alpha_3) & 0 & \frac{\epsilon_1}{\sqrt{2}} & \frac{\epsilon_1}{2}(\alpha_2-\alpha_3) \\
0 & 0 & 0 & \alpha_1 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_2 & 0 \\
0 & 0 & \frac{\epsilon_1}{2}(\alpha_2-\alpha_3) & 0 & \frac{1}{\sqrt{2}} & \frac{1}{2}(\alpha_2+\alpha_3)
\end{pmatrix}$$

- *Metaclass XXII:* $[31\,1\overline{1}]$ $\quad \epsilon_1 \in \{\pm 1\}$

$$\begin{pmatrix}
\alpha_1 & 0 & 0 & -\beta_1 & 0 & 0 \\
0 & \alpha_2 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{\epsilon_1}{\sqrt{2}} \\
0 & 0 & \frac{1}{2}(\alpha_2+\alpha_3) & 0 & \frac{\epsilon_1}{\sqrt{2}} & \frac{\epsilon_1}{2}(\alpha_2-\alpha_3) \\
\beta_1 & 0 & 0 & \alpha_1 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_2 & 0 \\
0 & 0 & \frac{\epsilon_1}{2}(\alpha_2-\alpha_3) & 0 & \frac{1}{\sqrt{2}} & \frac{1}{2}(\alpha_2+\alpha_3)
\end{pmatrix}$$

- *Metaclass XXIII:* $[31\,11]$ $\quad \epsilon_1 \in \{\pm 1\}$

$$\begin{pmatrix}
\alpha_3 & 0 & 0 & \alpha_4 & 0 & 0 \\
0 & \frac{1}{2}(\alpha_1+\alpha_2) & 0 & 0 & \frac{\epsilon_1}{2}(\alpha_1-\alpha_2) & \frac{\epsilon_1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \alpha_1 & 0 & \frac{\epsilon_1}{\sqrt{2}} & 0 \\
\alpha_4 & 0 & 0 & \alpha_3 & 0 & 0 \\
0 & \frac{\epsilon_1}{2}(\alpha_1-\alpha_2) & 0 & 0 & \frac{\epsilon_1}{2}(\alpha_1+\alpha_2) & \frac{1}{\sqrt{2}} \\
0 & 0 & 0 & 0 & 0 & \alpha_1
\end{pmatrix}$$

For each meta-class, $\alpha_i \in \mathbb{R}$, $\beta_i > 0$ for $i \in \{1, 2, \dots\}$ and conditions for signs $\epsilon_i \in \{-1, +1\}$ are given for each metaclass.

Proof. Let B be the 6×6 matrix $B = (\varepsilon^{IJ})_{IJ} = H_{(2)}$, where $\varepsilon^{IJ} = \varepsilon^{I_1 I_2 J_1 J_2}$ for $I, J \in O$, and $H_{(2)}$ is as in equation (32).

Claim 1. For any Segre type s in the list (18), there exists a non-empty finite set of invertible 6×6 matrices $\mathscr{S}_s \subset \mathbb{R}^{6\times 6}$ with the following property

$(*)$ If $\kappa \in \Omega^2_2(N)$ is such that ${}^{(2)}\kappa|_p = 0$ and $\kappa|_p$ has Segre type s, then

$$\kappa|_p \quad \sim \quad S \cdot V \cdot S^{-1}.$$

for some $S \in \mathscr{S}_s$, and a Jordan normal form matrix $V \in \mathbb{R}^{6\times 6}$ with Segre type s. (See Appendix B for the definition of Jordan normal form.)

To construct $\mathscr{S}_s$, let $s = \left[m_1 \cdots m_r\, k_1\overline{k_1} \cdots k_s\overline{k_s}\right]$ be a Segre type from the list (18), and let $\mathscr{W}_s \subset \mathbb{R}^{6\times 6}$ be the set of matrices of the form

$$W \quad = \quad \bigoplus_{j=1}^{r} \epsilon_j F_{m_j} \quad \oplus \quad \bigoplus_{j=1}^{s} F_{2k_j},$$

where $\epsilon_1, \dots, \epsilon_r \in \{\pm 1\}$ are such that *(i)* $\{\epsilon_j\}_{j=1}^r$ satisfy condition *(ii)* in Theorem B.3 and *(ii)* each $W \in \mathscr{W}_s$ has signature $(- - - + + +)$. It is clear that $\mathscr{W}_s$ is finite and computer algebra shows that $\mathscr{W}_s$ is not empty for any s. If $W \in \mathscr{W}_s$, then W and B are both symmetric matrices with spectrum $\{1, 1, 1, -1, -1, -1\}$ whence there exists an (orthogonal) $S \in \mathbb{R}^{6\times 6}$ such that

$$W \quad = \quad S^t \cdot B \cdot S. \tag{20}$$

Thus, for each $W \in \mathscr{W}_s$ we can find some $S \in \mathbb{R}^{6\times 6}$ such that equation (20) holds. Let us denote one such S by $S = S_W$, and let $\mathscr{S}_s = \{S_W \in \mathbb{R}^{6\times 6} : W \in \mathscr{W}_s\}$. Let us also note that $\mathscr{S}_s$ is not uniquely determined by s.

To show that $\mathscr{S}_s$ satisfies property $(*)$, let $\kappa \in \Omega^2_2(N)$ be such that ${}^{(2)}\kappa|_p = 0$ and $\kappa|_p$ has Segre type s. Moreover, in coordinates $\{x^i\}_{i=0}^3$ around p, let κ_I^J be components for $\kappa|_p$ as in equation (8). Then Theorem 2.1 implies that

$$\kappa_K^I \varepsilon^{KJ} = \kappa_K^J \varepsilon^{KI}, \quad I, J \in O.$$

For $A = (\kappa_I^J)_{IJ}$ we have $BA = A^t B$, so we can apply Theorem B.3, and there exists an invertible matrix $L \in \mathbb{R}^{6\times 6}$ such that

$$L^{-1} \cdot A \cdot L = V, \tag{21}$$

$$L^t \cdot B \cdot L = W, \tag{22}$$

where V is a Jordan normal form matrix with the same Segre type as $\kappa|_p$ and $W \in \mathscr{W}_s$. Now there exists an $S \in \mathscr{S}_s$ such that $W = S^t \cdot B \cdot S$ whence

$$A = (SL^{-1})^{-1} \cdot (S \cdot V \cdot S^{-1}) \cdot (SL^{-1}), \tag{23}$$

$$B = (SL^{-1})^t \cdot B \cdot (SL^{-1}) \tag{24}$$

and $\kappa|_p \sim S \cdot V \cdot S^{-1}$ follows by Proposition A.1 in Appendix A.

If $S \in \mathbb{R}^{6\times 6}$ is one solution to equation (20), then the set of all solutions is given by $\{\Lambda S \in \mathbb{R}^{6\times 6} : \Lambda^t \cdot B \cdot \Lambda = B\}$, and each solution typically gives rise to a different normal form for the metaclass. To complete the proof we need to go through all 23 Segre types, and for each Segre type s, we compute $S \cdot V \cdot S^{-1}$ for all $S \in \mathscr{S}_s$ (for a suitable choice of S and $\mathscr{S}_s$) and for all Jordan normal form matrices V with Segre type s. The choice of S and $\mathscr{S}_s$ are chosen so that normal forms on the theorem formulation correspond to the normal forms in [SWW10] via the correspondence in Proposition 2.5 with $g = \mathrm{diag}(1, 1, 1, 1)$.

To show the last claim for Metaclasses I, II, …, VI, VII, we need to show that the conjugations by Hodge star operators can be replaced by coordinate transformations and by possibly redefining the constants that appear in the normal form matrices. If $\{x^i\}_{i=0}^3$ are coordinates where $\kappa|_p \sim V$ holds, let $\{\widetilde{x}^i\}_{i=0}^3$ be coordinates determined by $\widetilde{x}^i = J^i_j x^j$ for a suitable 4×4 matrix $J = (J^i_j)_{ij}$. If $g_1 = \mathrm{diag}(1, 1, 1, 1)$ and $g_2 = \mathrm{diag}(1, -1, -1, 1)$ are metrics as in Definition 3.1 then suitable choices for J are

Metaclass	I	II	III	IV	V	VI	VII
Conjugation by $*_{g_1}$	J_1	J_2	J_3	J_1	J_2	J_1	Id
Conjugation by $*_{g_2}$	J_1	J_2	J_2	J_1	J_2	J_1	Id

where $J_1 = \mathrm{diag}(-1, 1, 1, 1)$ and

$$J_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad \square$$

Appendix A. Proposition A.1

In this appendix we state and prove Proposition A.1, which gives two alternative descriptions for $\kappa|_p \sim V$ in Definition 3.1.

Proposition A.1. *Suppose N is a 4-dimensional manifold, $\kappa \in \Omega^2_2(N)$, $p \in N$ and $V \in \mathbb{R}^{6\times 6}$. Then the following conditions are equivalent*

(i) $\kappa|_p \sim V$.

(ii) There are coordinates $\{x^i\}_{i=0}^3$ *around* p *and an* $\alpha \in \{1,2,3\}$ *such that*

$$(\kappa_I^J)_{IJ} = H_{(\alpha)} \cdot V \cdot H_{(\alpha)}, \tag{25}$$

where κ_I^J *are components that represent* $\kappa|_p$ *in coordinates* $\{x^i\}_{i=0}^3$ *as in equation* (8)*, and* $H_{(1)}$*,* $H_{(2)}$*,* $H_{(3)}$ *are the matrices in equations* (31)*–*(32) *in Appendix A.*

(iii) There are coordinates $\{x^i\}_{i=0}^3$ *around* p *and there exists an invertible* $S \in \mathbb{R}^{6\times 6}$ *such that*

$$(\kappa_I^J)_{IJ} = S^{-1} \cdot V \cdot S, \tag{26}$$

$$B = S^t \cdot B \cdot S, \tag{27}$$

where B *is the* 6×6 *matrix* $B = (\varepsilon^{IJ})_{IJ} = H_{(2)}$.

Proof. Equivalence *(i)* $\Leftrightarrow$ *(ii)* follows since $H_{(2)}$ and $-H_{(3)}$ are matrix representations of $*_g$ in the basis (7) for metrics $g = \operatorname{diag}(1,1,1,1)$ and $g = \operatorname{diag}(1,-1,-1,1)$, respectively. Implication *(ii)* $\Rightarrow$ *(iii)* follows by taking $S = H_{(\alpha)}$. For implication *(iii)* $\Rightarrow$ *(ii)*, let A be the 6×6 matrix $A = (\kappa_I^J)_{IJ}$, let $\{T_I^J : I, J \in O\}$ be the array of components such that $(T_I^J)_{IJ} = S^{-1}$, and for each $J \in O$ let $T^J \in \Lambda_p^2(N)$ be defined by

$$T^J = T_I^J dx^I. \tag{28}$$

Equation (27) implies that $B = S^{-t} \cdot B \cdot S^{-1}$. Thus $\{T^J : J \in O\}$ satisfy the assumptions in Proposition A.2 whence there exist linearly independent covectors $\{\xi^i\}_{i=0}^3$ in $\Lambda_p^1(N)$ such that equation (30) holds for some $\alpha \in \{0,1,2,3\}$. Around p, let $\{\widetilde{x}^i\}_{i=0}^3$ be coordinates defined as $\widetilde{x}^i = \xi^i\left(\frac{\partial}{\partial x^b}\Big|_p\right) x^b$. To see that $\{\widetilde{x}^i\}_{i=0}^3$ are coordinates it suffices to show that $\big(dx^i(u_j)\big)_{ij}$ is the inverse matrix to $\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right)_{ij}$ when $\{u_i\}_{i=0}^3$ is a dual basis to $\{\xi^i\}_{i=0}^3$. Thus $\xi^i = d\widetilde{x}^i|_p$ and equations (30), (28) and $d\widetilde{x}^I = \frac{\partial \widetilde{x}^I}{\partial x^L} dx^L$ imply that $T_I^J = Y_{(\alpha)K}^J \frac{\partial \widetilde{x}^K}{\partial x^I}$. Equation (26) further implies that $A \cdot S^{-1} = S^{-1} \cdot V$ and by equation (10),

$$\widetilde{\kappa}_I^L Y_{(\alpha)L}^J = Y_{(\alpha)I}^L V_{b(L)b(J)}, \quad I, J \in O.$$

Since $H_{(\alpha)}^2 = \mathrm{Id}$ it follows that $(\widetilde{\kappa}_I^J)_{IJ} = H_{(\alpha)} \cdot V \cdot H_{(\alpha)}$ where $\alpha \in \{0,1,2,3\}$, and part *(ii)* follows. □

Proposition A.2. *Suppose* $T^I \in \Lambda_p^2(N)$ *for all* $I \in O$ *on a 4-manifold* N*, where* O *is as in Section 2.3 and* $p \in N$*. Moreover, suppose that*

$$T^I \wedge T^J = \varepsilon^{IJ}\omega, \quad I, J \in O \tag{29}$$

for some $\omega \in \Lambda_p^4(N)\backslash\{0\}$*. Then there exists linearly independent* $\xi_0, \ldots, \xi_3 \in \Lambda_p^1(N)$ *and an* $\alpha \in \{0,1,2,3\}$ *such that*

$$T^J = Y_{(\alpha)I}^J \xi^I, \quad J \in O, \tag{30}$$

where $\xi^I = \xi^{I_1} \wedge \xi^{I_2}$ *and* $Y_{(\alpha)I}^J$ *are components such that* $(Y_{(\alpha)I}^J)_{IJ} = H_{(\alpha)}$ *for one of the* 6×6 *matrices*

$$H_{(0)} = -\mathrm{Id}, \qquad H_{(1)} = \mathrm{Id}, \tag{31}$$

$$H_{(2)} = \begin{pmatrix} & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \end{pmatrix}, \quad H_{(3)} = \begin{pmatrix} & & & 1 & & \\ & & & & 1 & \\ & & & & & -1 \\ 1 & & & & & \\ & 1 & & & & \\ & & -1 & & & \end{pmatrix}. \tag{32}$$

Proof. Let us first note that equation (29) implies that T^I is non-zero for each $I \in O$. Let g be an auxiliary positive definite Riemann metric on N. Then

$$\langle u, v\rangle \quad = \quad *_g(u\wedge v), \quad u, v \in \Lambda^2_p(N)$$

defines an indefinite inner product in $\Lambda^2_p(N)$ of signature $(+++---)$ [Har91]. For a vector subspace $W \subset \Lambda^2_p(N)$, we denote the *orthogonal complement* (with respect to $\langle\cdot,\cdot\rangle$) by $W^\perp$ [O'N83, p. 49].

Claim 1. There exists linearly independent covectors $\xi^0, \xi^1, \xi^2 \in \Lambda^1_p(N)$ such that

$$T^{0i} \quad = \quad \xi^0 \wedge \xi^i, \quad i \in \{1, 2\}. \tag{33}$$

In four dimensions, the Plucker identities states that a $q \in \Lambda^2_p(N)$ can be written as $q = a \wedge b$ for some $a, b \in \Lambda^1_p(N)$ if and only if $q \wedge q = 0$ [Coh05, p. 184]. Thus equation (29) implies that there exist $\xi_0, \xi_1 \in \Lambda^1_p(N)$ such that

$$T^{01} = \xi^0 \wedge \xi^1. \tag{34}$$

Since $T^{01} \neq 0$, covectors ξ^0 and ξ^1 are linearly independent. Let $\xi^2, \xi^3 \in \Lambda^1_p(N)$ be such that $\{\xi^i\}_{i=0}^3$ is a basis for $\Lambda^1_p(N)$. For $W = \operatorname{span}\{T^{01}\}$ we then have $\dim W^\perp = 5$ and

$$W^\perp \quad = \quad \operatorname{span}\{\{\xi^0 \wedge \xi^r\}_{r=1}^3,\ \{\xi^1 \wedge \xi^s\}_{s=2}^3\}.$$

Since $T^{02} \in W^\perp$ we have

$$T^{02} \quad = \quad A\xi^0 \wedge \xi^1 + \xi^0 \wedge \zeta^0 + \xi^1 \wedge \zeta^1, \tag{35}$$

for some $A \in \mathbb{R}$ and $\zeta^0, \zeta^1 \in \operatorname{span}\{\xi^i\}_{i=2}^3$. From $T^{02} \wedge T^{02} = 0$, it follows that $\xi^0 \wedge \xi^1 \wedge \zeta^0 \wedge \zeta^1 = 0$. Thus covectors $\xi^0, \xi^1, \zeta^0, \zeta^1$ are linearly dependent and there are constants C_i such that

$$C_1\xi^0 + C_2\xi^1 + C_3\zeta^0 + C_4\zeta^1 \quad = \quad 0, \tag{36}$$

and all C_1, C_2, C_3, C_4 are not zero. Since $C_1 = C_2 = 0$, we can not have $C_3 = C_4 = 0$. If $C_3 \neq 0$, then $\zeta^0 = -\frac{C_4}{C_3}\zeta^1$ and equations (34) and (35) yield

$$T^{01} = \left(\xi^1 - \frac{C_4}{C_3}\xi^0\right) \wedge (-\xi^0), \quad T^{02} = \left(\xi^1 - \frac{C_4}{C_3}\xi^0\right) \wedge \left(\zeta^1 - A\xi^0\right).$$

If $\zeta^1 = 0$ then $\zeta^0 = 0$ whence equation (35) implies that $T^{02} = AT^{01}$ and $A \neq 0$. Writing out $AT^{01} \wedge T^{23} = T^{02} \wedge T^{23}$ using equation (29) gives a contradiction, so $\zeta^1 \neq 0$. Hence covectors $\xi^1 - \frac{C_4}{C_3}\xi^0, -\xi^0$ and $\zeta^1 - A\xi^0$ are linearly independent and Claim 1 follows. The case $C_4 \neq 0$ follows similarly.[1]

Claim 2. There exists a basis $\{\xi^i\}_{i=0}^3$ for $\Lambda^1_p(N)$ such that equations (33) hold and

$$T^{03} \quad = \quad \xi^0 \wedge \zeta + D\xi^1 \wedge \xi^2 \tag{37}$$

for some $D \in \mathbb{R}$ and $\zeta \in \operatorname{span}\{\xi^i\}_{i=1}^3$.

If $\xi^0, \xi^1, \xi^2 \in \Lambda^1_p(N)$ are as in Claim 1, then there exists a $\xi^3 \in \Lambda^1_p(N)$ such that $\{\xi^i\}_{i=0}^3$ is a basis for $\Lambda^1_p(N)$. For $W = \operatorname{span}\{T^{01}, T^{02}\}$, we then have $\dim W^\perp = 4$ and

$$W^\perp \quad = \quad \operatorname{span}\{\xi^1 \wedge \xi^2,\ \{\xi^0 \wedge \xi^i\}_{i=1}^3\}.$$

[1] Explicitly,

$$T^{01} = \left(\xi^0 - \frac{C_3}{C_4}\xi^1\right) \wedge (\xi^1), \quad T^{02} = \left(\xi^0 - \frac{C_3}{C_4}\xi^1\right) \wedge \left(\zeta^0 + A\xi^1\right).$$

Since $T^{03} \in W^\perp$ there exists a $\zeta \in \operatorname{span}\{\xi^1, \xi^2, \xi^3\}$ and a $D \in \mathbb{R}$ such that equation (37) holds.

The proof now divides into two cases depending on D in Claim 2.

Claim 3. If Claim 2 holds with $D = 0$, then there are linearly independent $\xi^0, \ldots, \xi^3 \in \Lambda^1_p(N)$ and a $\tau \in \{\pm 1\}$ such that

$$T^{0i} = \xi^0 \wedge \xi^i, \quad i \in \{1,2,3\}, \tag{38}$$

$$T^{12} = \tau\xi^1 \wedge \xi^2, \tag{39}$$

$$T^{23} = \tau\xi^2 \wedge \xi^3, \tag{40}$$

$$T^{31} = \tau\xi^3 \wedge \xi^1. \tag{41}$$

The proof is divided into four steps. In Step 1, let us show that there are linearly independent $\{\xi^i\}_{i=0}^3$ such that equations (38) hold. Since $D = 0$ in Claim 2, equations (38) hold by setting $\xi^3 = \zeta$. Therefore we only need to show that $\{\xi^i\}_{i=0}^3$ are linearly independent. If there are constants $C_0, \ldots, C_3 \in \mathbb{R}$ such that $\sum_{i=0}^3 C_i\xi^i = 0$, then

$$C_1T^{01} + C_2T^{02} + C_3T^{03} = 0.$$

Thus $C_1T^{01} \wedge T^{23} = 0$ so $C_1 = 0$ by equation (29). Similarly we obtain $C_2 = C_3 = 0$. Thus $C_0\xi^0 = 0$, so $C_0 = 0$, and $\{\xi^i\}_{i=0}^3$ are linearly independent.

In Step 2, let us show that there exists a $\tau \in \{\pm 1\}$ and linearly independent $\{\xi^i\}_{i=0}^3$ such that equations (38)–(39) hold. By Step 1, there are linearly independent $\{\xi^i\}_{i=0}^3$ such that equations (38) hold. We know that $T^{12} \in \operatorname{span}\{T^{01}, T^{02}\}^\perp$. Hence

$$T^{12} = \xi^0 \wedge \zeta + E\xi^1 \wedge \xi^2 \tag{42}$$

where $\zeta \in \operatorname{span}\{\xi^i\}_{i=1}^3$ and $E \in \mathbb{R}$. Equations (38), (29) and (42) imply that $\omega = T^{03} \wedge T^{12} = E\xi^0 \wedge \xi^1 \wedge \xi^2 \wedge \xi^3$, so $E \neq 0$. Let $\tau = \operatorname{sgn} E$. Since $T^{12} \wedge T^{12} = 0$, it follows that $\xi^0, \xi^1, \xi^2, \zeta$ are linearly dependent and there are constants $C_0, \ldots, C_3$ such that

$$C_0\xi^0 + C_1\xi^1 + C_2\xi^2 + C_3\zeta = 0$$

and all $C_0, \ldots, C_3$ are not zero. It is clear that $C_0 = 0$. Since $C_3 = 0$ is not possible, there are constants $A, B \in \mathbb{R}$ such that

$$T^{12} = \xi^0 \wedge \left(A\xi^1 + B\xi^2\right) + E\xi^1 \wedge \xi^2.$$

Thus

$$\begin{aligned}
T^{01} &= \left(\frac{1}{\sqrt{|E|}}\xi^0\right) \wedge \left(\sqrt{|E|}\xi^1 + \frac{\tau B}{\sqrt{|E|}}\xi^0\right), \\
T^{02} &= \left(\frac{1}{\sqrt{|E|}}\xi^0\right) \wedge \left(\sqrt{|E|}\xi^2 - \frac{\tau A}{\sqrt{|E|}}\xi^0\right), \\
T^{03} &= \left(\frac{1}{\sqrt{|E|}}\xi^0\right) \wedge \left(\sqrt{|E|}\xi^3\right), \\
T^{12} &= \tau\left(\sqrt{|E|}\xi^1 + \frac{\tau B}{\sqrt{|E|}}\xi^0\right) \wedge \left(\sqrt{|E|}\xi^2 - \frac{\tau A}{\sqrt{|E|}}\xi^0\right).
\end{aligned}$$

Since the four covectors inside the parentheses are linearly independent, Step 2 follows.

In Step 3, let us show that there exists a $\tau \in \{\pm 1\}$ and linearly independent $\{\xi^i\}_{i=0}^3$ such that equations (38)–(40) hold. By Step 2, there exists a $\tau \in \{\pm 1\}$ and linearly independent covectors $\{\xi^i\}_{i=0}^3$ such that (38)–(39) hold. Since $T^{23} \in \operatorname{span}\{T^{12}, T^{02}, T^{03}\}^\perp$ we have

$$T^{23} = \xi^0 \wedge \zeta + F\xi^2 \wedge \xi^3$$

for some $\zeta \in \operatorname{span}\{\xi^a\}_{a=1}^2$ and $F \in \mathbb{R}$. Writing out $T^{01} \wedge T^{23} = T^{03} \wedge T^{12}$ shows that $F = \tau$. Since $T^{23} \wedge T^{23} = 0$ there are constants $C_0, \ldots, C_3$ such that

$$C_0\xi^0 + C_1\xi^2 + C_2\xi^3 + C_3\zeta = 0,$$

and all $C_0, \ldots, C_3$ are not zero. Now $C_0 = 0$ and $C_2 = 0$. Since $C_3 \neq 0$ is not possible, it follows that $\zeta = C\xi^2$ for some $C \in \mathbb{R}$. Thus $T^{23} = \tau\xi^2 \wedge (\xi^3 - \tau C\xi^0)$, and Step 3 follows by rewriting $T^{01}, T^{02}, T^{03}, T^{12}, T^{23}$ and checking linear independence as in Step 2.

In Step 4, let us show that there exists a $\tau \in \{\pm 1\}$ and linearly independent $\{\xi^i\}_{i=0}^3$ such that equations (38)–(41) hold. By Step 3, there exist a $\tau \in \{\pm 1\}$ and linearly independent covectors $\{\xi^i\}_{i=0}^3$ such that equations (38)–(40) hold. Since $T^{31} \in \operatorname{span}\{T^{12}, T^{01}, T^{03}, T^{23}\}^\perp$ it follows that

$$T^{31} = A\xi^3 \wedge \xi^1 + B\xi^0 \wedge \xi^2$$

for some $A, B \in \mathbb{R}$. Writing out $T^{02} \wedge T^{31} = T^{01} \wedge T^{23}$ gives $A = \tau$ and writing out $T^{31} \wedge T^{31} = 0$ gives $B = 0$. This completes the proof of Claim 3.

Claim 4. Suppose Claim 2 holds with $D \neq 0$ and let $\sigma = \operatorname{sgn} D$. Then there are linearly independent $\xi^0, \ldots, \xi^3 \in \Lambda^1_p(N)$ such that

$$T^{01} = \xi^2 \wedge \xi^3, \tag{43}$$
$$T^{02} = \xi^3 \wedge \xi^1, \tag{44}$$
$$T^{03} = \sigma\xi^1 \wedge \xi^2, \tag{45}$$
$$T^{12} = \sigma\xi^0 \wedge \xi^3, \tag{46}$$
$$T^{23} = \xi^0 \wedge \xi^1, \tag{47}$$
$$T^{31} = \xi^0 \wedge \xi^2. \tag{48}$$

As the proof of Claim 3, the proof is divided into four steps. In Step 1, let us show that there are linearly independent $\{\xi^i\}_{i=1}^3$ such that equations (43)–(45) hold. Let $\{\xi^i\}_{i=0}^3$, ζ and $D \neq 0$ be as in Claim 2. Then $T^{03} \wedge T^{03} = 0$ implies that $\zeta = A\xi^1 + B\xi^2$ for some $A, B \in \mathbb{R}$, and

$$\begin{aligned}
T^{01} &= \left(-\sqrt{|D|}\xi^1 - \frac{\sigma B}{\sqrt{|D|}}\xi^0\right) \wedge \left(\frac{1}{\sqrt{|D|}}\xi^0\right), \\
T^{02} &= \left(\frac{1}{\sqrt{|D|}}\xi^0\right) \wedge \left(\sqrt{|D|}\xi^2 - \frac{\sigma A}{\sqrt{|D|}}\xi^0\right), \\
T^{03} &= \sigma\left(\sqrt{|D|}\xi^2 - \frac{\sigma A}{\sqrt{|D|}}\xi^0\right) \wedge \left(-\sqrt{|D|}\xi^1 - \frac{\sigma B}{\sqrt{|D|}}\xi^0\right).
\end{aligned}$$

Step 1 follows since the covectors in the parentheses are linearly independent.

In Step 2, let us show that there are linearly independent $\{\xi^i\}_{i=0}^3$ such that equations (43)–(46) hold. By Step 1, there are linearly independent $\{\xi^i\}_{i=1}^3$ such that equations (43)–(45) hold. We know that $T^{12} \in \operatorname{span}\{T^{01}, T^{02}\}^\perp$. Hence

$$T^{12} = \xi^3 \wedge \zeta + E\xi^1 \wedge \xi^2$$

where $\zeta \in \Lambda^1_p(N)$ and $E \in \mathbb{R}$. Writing out $T^{03} \wedge T^{12} \neq 0$ and $T^{12} \wedge T^{12} = 0$ shows that $\xi^1, \xi^2, \xi^3, \zeta$ are linearly independent and $E = 0$. Step 2 follows by setting $\xi^0 = -\sigma\zeta$.

In Step 3, let us show that there are linearly independent $\{\xi^i\}_{i=0}^3$ such that equations (43)–(47) hold. By Step 2, there exist linearly independent $\{\xi^i\}_{i=0}^3$ such that (43)–(46) hold. Since $T^{23} \in \operatorname{span}\{T^{12}, T^{02}, T^{03}\}^\perp$, it follows that

$$T^{23} = \xi^1 \wedge (A\xi^0 + B\xi^3) + E\xi^2 \wedge \xi^3$$

for some $A, B, E \in \mathbb{R}$. Writing out $T^{01} \wedge T^{23} = T^{03} \wedge T^{12}$ and $T^{23} \wedge T^{23} = 0$ gives $A = -1$ and $E = 0$. Thus

$$T^{23} = (\xi^0 - B\xi^3) \wedge \xi^1,$$

and Step 3 follows since T^{12} can be rewritten as $T^{12} = \sigma(\xi^0 - B\xi^3) \wedge \xi^3$.

In Step 4, let us show that there are linearly independent $\{\xi^i\}_{i=0}^3$ such that equations (43)–(48) hold. By Step 3, there exist linearly independent covectors $\{\xi^i\}_{i=0}^3$ such that (43)–(47) hold. Since $T^{31} \in \operatorname{span}\{T^{01}, T^{03}, T^{12}, T^{23}\}^\perp$ we have

$$T^{31} = A\xi^3 \wedge \xi^1 + B\xi^0 \wedge \xi^2$$

for some $A, B \in \mathbb{R}$. Writing out $T^{31} \wedge T^{02} = T^{01} \wedge T^{23}$ and $T^{31} \wedge T^{31} = 0$ gives $B = 1$ and $A = 0$, so equation (48) holds and Step 4 follows. This completes the proof of Claim 4.

When Claim 3 holds, then equation (30) holds for covectors $\{\tau\xi^0, \xi^1, \xi^2, \xi^3\}$ and $\alpha = 0$ when $\tau = -1$ and $\alpha = 1$ when $\tau = 1$. When Claim 4 holds, then equation (30) holds with $\alpha = 2$ when $\sigma = 1$ and $\alpha = 3$ when $\sigma = -1$. $\square$

Appendix B. Normal form for a H-selfadjoint matrix

The Jordan normal form theorem (Theorem B.1) is a fundamental theorem in linear algebra. In this appendix we formulate Theorem B.3 which extends this result to two matrices that are suitably compatible. The result is known as the *canonical form of an H-selfadjoint matrix.* The result and its proof can be found in [LR05, Theorem 12.2].

First we define the block matrices that appear in the Jordan normal form theorem for real matrices [LR05, Theorem 2.2]. For $m \in \{1, 2, \ldots\}$, $\lambda, \sigma \in \mathbb{R}$ and $\tau > 0$ let

$$R_m(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix} \in \mathbb{R}^{m\times m},$$

$$C_{2m}(\sigma \pm i\tau) = \begin{pmatrix} \sigma & \tau & 1 & 0 & & & & \\ -\tau & \sigma & 0 & 1 & & & & \\ & & \sigma & \tau & 1 & 0 & & \\ & & -\tau & \sigma & 0 & 1 & & \\ & & & & \ddots & & \ddots & \\ & & & & & \ddots & 1 & 0 \\ & & & & & \ddots & 0 & 1 \\ & & & & & & \sigma & \tau \\ & & & & & & -\tau & \sigma \end{pmatrix} \in \mathbb{R}^{2m\times 2m}.$$

Moreover, let $F_1 = (1)$ and for $m \geq 2$, let F_m be the *standard involutory permutation* matrix

$$F_m \quad = \quad \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \in \mathbb{R}^{m\times m}. \tag{49}$$

For square matrices $M_1, \ldots, M_k$, we define

$$M_1 \oplus \cdots \oplus M_k \quad = \quad \begin{pmatrix} M_1 & & \\ & \ddots & \\ & & M_k \end{pmatrix}. \tag{50}$$

The next theorem is the *Jordan normal form theorem* with the ordering in equation (52) being a consequence of Proposition B.2. We say that a matrix $A \in \mathbb{R}^{n\times n}$ is in *Jordan normal form* if Theorem B.1 holds with $L = \mathrm{Id}$.

Theorem B.1. *Suppose $A \in \mathbb{R}^{n\times n}$. Then there exists an invertible matrix $L \in \mathbb{R}^{n\times n}$ such that*

$$L^{-1}AL \quad = \quad \bigoplus_{j=1}^{r} R_{m_j}(\lambda_j) \quad \oplus \quad \bigoplus_{j=1}^{s} C_{2k_j}(\sigma_j \pm i\tau_j), \tag{51}$$

for some $r, s \geq 0$, $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, $\sigma_1, \ldots, \sigma_s \in \mathbb{R}$, $\tau_1, \ldots, \tau_s > 0$ and

$$m_1 \geq \cdots \geq m_r \geq 1, \quad k_1 \geq \cdots \geq k_s \geq 1. \tag{52}$$

Moreover, suppose that $\widetilde{L}$ is another $n\times n$ matrix such that equations (51) *and* (52) *hold for block matrices $(R_{\widetilde{m}_j}(\widetilde{\lambda}_j))_{j=1}^{\widetilde{r}}$ and $(C_{2\widetilde{k}_j}(\widetilde{\sigma}_j \pm i\widetilde{\tau}_j))_{j=1}^{\widetilde{s}}$. Then $\widetilde{r} = r$, $\widetilde{s} = s$ and $(R_{\widetilde{m}_j}(\widetilde{\lambda}_j))_{j=1}^{r}$ is a permutation of $(R_{m_j}(\lambda_j))_{j=1}^{r}$ and $(C_{2\widetilde{k}_j}(\widetilde{\sigma}_j \pm i\widetilde{\tau}_j))_{j=1}^{s}$ is a permutation of $(C_{2k_j}(\sigma_j \pm i\tau_j))_{j=1}^{s}$. In particular, $\widetilde{m}_j = m_j$ for $j = 1, \ldots, r$ and $\widetilde{k}_j = k_j$ for $j = 1, \ldots, s$.*

The next proposition shows that the blocks in $M_1 \oplus \cdots \oplus M_k$ can be permutated into any order using a similarity transformation [Fie86, p. 31].

Proposition B.2. *Suppose*

$$A \quad = \quad M_1 \oplus \cdots \oplus M_k,$$

where $M_1, \ldots, M_k$ are real square matrices, and suppose that π is a permutation of $\{1, 2, \ldots, k\}$. Then there exists a real orthogonal matrix P such that

$$P^{-1}AP \quad = \quad M_{\pi(1)} \oplus \cdots \oplus M_{\pi(k)}.$$

For example, if $M_1 \in \mathbb{R}^{n\times n}$ and $M_2 \in \mathbb{R}^{m\times m}$ then $P^{-1} \cdot (M_1 \oplus M_2) \cdot P = M_2 \oplus M_1$ for $P = \begin{pmatrix} 0_{n\times m} & I_{n\times n} \\ I_{m\times m} & 0_{m\times n} \end{pmatrix}$, where $0_{a\times b}$ is the $a \times b$ zero matrix, and $I_{a\times a}$ is the $a\times a$ identity matrix.

Theorem B.3. *Suppose $A, B \in \mathbb{R}^{n\times n}$ are matrices such that*

$$B = B^t, \quad \det B \neq 0, \quad BA = A^t B.$$

Then there exists an invertible $n \times n$ matrix L such that

$$\begin{aligned} L^{-1}AL \quad &= \quad \bigoplus_{j=1}^{r} R_{m_j}(\lambda_j) \quad \oplus \quad \bigoplus_{j=1}^{s} C_{2k_j}(\sigma_j \pm i\tau_j), \\ L^t BL \quad &= \quad \bigoplus_{j=1}^{r} \epsilon_j F_{m_j} \quad \oplus \quad \bigoplus_{j=1}^{s} F_{2k_j}, \end{aligned}$$

where $r, s \geq 0$, $\lambda_1, \dots, \lambda_r \in \mathbb{R}$, $\sigma_1, \dots, \sigma_s \in \mathbb{R}$, $\tau_1, \dots, \tau_s > 0$ *and* $\epsilon_1, \dots, \epsilon_r \in \{\pm 1\}$. *Moreover,*

(i) $m_1 \geq \cdots \geq m_r \geq 1$ *and* $k_1 \geq \cdots \geq k_s \geq 1$,
(ii) if $m_a = m_{a+1} = \cdots = m_{a+d}$ *for some* $1 \leq a < a + d \leq r$, *then*

$$\epsilon_a \leq \epsilon_{a+1} \leq \cdots \leq \epsilon_{a+d}.$$

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